Vectors Cheat Sheet

This chapter builds upon the basic vectors you covered in Pure Year 1. We will extend our knowledge to 3 dimensions and learn to solve a greater range of problems.

It is important to realise that working in 3-dimensions is very similar to working in 2 dimensions. The techniques you learnt in Pure Year 1 can also be applied to 3 dimensions.

3D Coordinates

With 3 dimensions, we now have a third axis, known as the z axis. As a result,

- Points in 3-dimensions are written as (x, y, z).

Representing vectors

We can represent 3D vectors using column notation or the unit vectors along the x, y and z axes.

- The unit vectors, denoted i, j, k are: $i = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $j = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $k = \begin{pmatrix}0\\0\\1\end{pmatrix}$
- For any vector in 3D, $ai + bj + ck = \begin{pmatrix}a\\b\\c\end{pmatrix}$

Using unit vectors

Column notation

Finding distances

The use of Pythagoras Theorem to find distances in 3D is quite similar to its use in 2D.

- The distance from the origin to the point (x, y, z) is $\sqrt{x^2 + y^2 + z^2}$.
- The magnitude of a vector a $= \begin{pmatrix}x\\y\\z\end{pmatrix}$ is simply the distance from the origin to the point (x, y, z) and therefore is also equal to $\sqrt{x^2 + y^2 + z^2}$
- The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Example 1: Consider the vectors $a = \begin{pmatrix}1\\2\\4\end{pmatrix}$ and $b = \begin{pmatrix}3\\-1\\2\end{pmatrix}$. Find $4\boldsymbol{a} + 8\boldsymbol{b}$, writing your answer in both column and **ijk** notation.

$$4\boldsymbol{a} = 4\begin{pmatrix}1\\2\\4\end{pmatrix} = \begin{pmatrix}4\times1\\4\times2\\4\times4\end{pmatrix} = \begin{pmatrix}4\\8\\16\end{pmatrix}$$

$$8\boldsymbol{b} = 8\begin{pmatrix}3\\-1\\2\end{pmatrix} = \begin{pmatrix}8\times3\\8\times-1\\8\times2\end{pmatrix} = \begin{pmatrix}24\\-8\\16\end{pmatrix}$$

$\therefore 4\boldsymbol{a} + 8\boldsymbol{b} = \begin{pmatrix}4\\8\\16\end{pmatrix} + \begin{pmatrix}24\\-8\\16\end{pmatrix} = \begin{pmatrix}28\\0\\32\end{pmatrix}$ in column notation.

In **ijk** notation, this is $28\boldsymbol{i} + 0\boldsymbol{j} + 32\boldsymbol{k} = 28\boldsymbol{i} + 32\boldsymbol{k}$.

In general, you are free to use whichever notation you like. Column notation tends to be easier to follow and less tedious to write, however.

Unit vectors

A unit vector is any vector of magnitude 1. If we have a vector a, then we can construct a unit vector in the direction of a, denoted as â, using the formula:

- $\hat{a} = \frac{a}{|a|}$, where $|a|$ is the magnitude of a.

Angles between axes

You can work out the angle between a vector a $= (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$ and any of the axes using the following formulae:

- $\cos\theta_x = \frac{x}{|a|}$
- $\cos\theta_y = \frac{y}{|a|}$
- $\cos\theta_z = \frac{z}{|a|}$

$\theta_x, \theta_y, \theta_z$ represent the angles **a** makes with the x, y and z axes respectively.

Example 2: Given that $\boldsymbol{c} = \begin{pmatrix}3\\-4\\-2\end{pmatrix}$,

a) Find the unit vector in the direction of $\boldsymbol{c}$.
b) Find the angle $\boldsymbol{c}$ makes with the x, y and z axes.

a) $\hat{c} = \frac{c}{|c|}$

$|c| = \sqrt{(3)^2 + (-4)^2 + (-2)^2} = \sqrt{29}$

$\therefore \hat{c} = \frac{1}{\sqrt{29}}\begin{pmatrix}3\\-4\\-2\end{pmatrix}$

b) $\cos\theta_x = \frac{x}{|c|} = \frac{3}{\sqrt{29}} \Rightarrow \theta_x = \cos^{-1}\left(\frac{3}{\sqrt{29}}\right) = 56.1°$ = angle with x-axis.

$\cos\theta_y = \frac{y}{|c|} = \frac{-4}{\sqrt{29}} \Rightarrow \theta_y = \cos^{-1}\left(\frac{-4}{\sqrt{29}}\right) = 138°$ = angle with y-axis.

$\cos\theta_z = \frac{z}{|c|} = \frac{-2}{\sqrt{29}} \Rightarrow \theta_z = \cos^{-1}\left(\frac{-2}{\sqrt{29}}\right) = 111.8°$ = angle with z-axis.

Solving geometric problems

You need to be able to apply everything from AS and A2 vectors to geometric problems in 3 dimensions. The following facts tend to be useful for such questions:

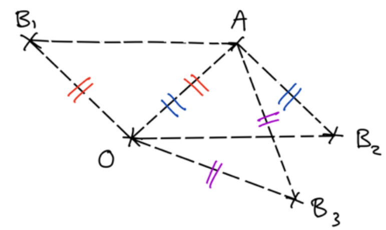
- If we have $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$, then we can compare coefficients on both sides of the equation giving us $a = u, b = v, c = w$.
- If the point P divides the line segment AB in the ratio $\lambda : \mu$, then

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{\lambda}{\lambda + \mu}\overrightarrow{AB}$$

Drawing a big diagram is always helpful for geometric problems. We will go through an example showcasing the typical style of such questions.

Example 4: The points A and B have position vectors **10i-23j+10k** and **pi+14j-22k** respectively, relative to a fixed origin O, where p is a constant. Given that ΔOAB is isosceles, find three possible positions of the point B.

We start with a diagram. This helps us to understand why there are three possible positions for B.



The first possible position is when OB and OA are equal, the second when OA and AB are equal and the third is when sides OB and AB are equal. To find each position vector, we can equate the lengths of the two sides that are equal.

$B = \begin{pmatrix}p\\14\\-22\end{pmatrix}$. So $\overrightarrow{AB} = \begin{pmatrix}p\\14\\-22\end{pmatrix} - \begin{pmatrix}10\\-23\\10\end{pmatrix} = \begin{pmatrix}p-10\\37\\-32\end{pmatrix}$

For B_1: $\overrightarrow{OB} = \overrightarrow{OA}$; $\sqrt{(p-10)^2 + 14^2 + (-22)^2} = \sqrt{(10)^2 + (-23)^2 + (10)^2} = 27$
$\Rightarrow p^2 + 680 = 27^2 \therefore p = \pm 7$ (squaring both sides and rearranging)

For B_2: $\overrightarrow{OA} = \overrightarrow{AB}$; $\sqrt{(10)^2 + (-23)^2 + (10)^2} = \sqrt{(p-10)^2 + (37)^2 + (-32)^2}$
$\Rightarrow 27^2 = (p-10)^2 + (37)^2 + (-32)^2$ (squaring both sides and rearranging)
$\Rightarrow 729 = p^2 - 20y + 2493$.
This results in a quadratic where the discriminant is less than 0, so there are no real solutions for p here.

For B_3: $\overrightarrow{OB} = \overrightarrow{AB}$; $\sqrt{(p)^2 + (14)^2 + (-22)^2} = \sqrt{(p-10)^2 + (37)^2 + (-32)^2}$
$\Rightarrow p^2 + 680 = (p-10)^2 + 2393$ (squaring both sides and collecting like terms)
$\Rightarrow 20p = 1813 \therefore p = \frac{1813}{20}$. (rearranging and finding p)

So our three possible position vectors of B are $\begin{pmatrix}7\\14\\-22\end{pmatrix}$, $\begin{pmatrix}-7\\14\\-22\end{pmatrix}$ and $\begin{pmatrix}\frac{1813}{20}\\14\\-22\end{pmatrix}$.