

# Vectors Cheat Sheet

This chapter builds upon the basic vectors you covered in Pure Year 1. We will extend our knowledge to 3 dimensions and learn to solve a greater range of problems.

It is important to realise that working in 3-dimensions is very similar to working in 2 dimensions. The techniques you learnt in Pure Year 1 can also be applied to 3 dimensions.

## 3D Coordinates

With 3 dimensions, we now have a third axis, known as the  $z$  axis. As a result,

- Points in 3-dimensions are written as  $(x, y, z)$ .

## Representing vectors

We can represent 3D vectors using column notation or the unit vectors along the  $x, y$  and  $z$  axes.

The unit vectors, denoted  $i, j, k$  are:

$$i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For any vector in 3D,  $ai + bj + ck = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Using unit vectors Column notation

## Finding distances

The use of Pythagoras Theorem to find distances in 3D is quite similar to its use in 2D.

- The distance from the origin to the point  $(x, y, z)$  is  $\sqrt{x^2 + y^2 + z^2}$ .
- The magnitude of a vector  $a = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$  is simply the distance from the origin to the point  $(x, y, z)$  and therefore is also equal to  $\sqrt{x^2 + y^2 + z^2}$
- The distance between the points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  is  $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ .

**Example 1:** Consider the vectors  $a = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$  and  $b = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ . Find  $4a + 8b$ , writing your answer in both column and  $ijk$  notation.

$$4a = 4 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 16 \end{pmatrix}$$

$$8b = 8 \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 16 \\ 32 \end{pmatrix}$$

$$\therefore 4a + 8b = \begin{pmatrix} 4 \\ 8 \\ 16 \end{pmatrix} + \begin{pmatrix} 24 \\ 16 \\ 32 \end{pmatrix} = \begin{pmatrix} 28 \\ 24 \\ 48 \end{pmatrix} \text{ in column notation.}$$

In  $ijk$  notation, this is  $12i + 24j + 48k$ .

In general, you are free to use whichever notation you like. Column notation tends to be easier to follow and less tedious to write, however.

## Unit vectors

A unit vector is any vector of magnitude 1. If we have a vector  $a$ , then we can construct a unit vector in the direction of  $a$ , denoted as  $\hat{a}$ , using the formula:

$$\hat{a} = \frac{a}{|a|}, \text{ where } |a| \text{ is the magnitude of } a.$$

## Angles between axes

You can work out the angle between a vector  $a = (xi + yj + zk)$  and any of the axes using the following formulae:

$$\begin{aligned} \cos \theta_x &= \frac{x}{|a|} \\ \cos \theta_y &= \frac{y}{|a|} \\ \cos \theta_z &= \frac{z}{|a|} \end{aligned}$$

$\theta_x, \theta_y, \theta_z$  represent the angles  $a$  makes with the  $x, y$  and  $z$  axes respectively.

**Example 2:** Given that  $c = \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$ ,

- Find the unit vector in the direction of  $c$ .
- Find the angle  $c$  makes with the  $x, y$  and  $z$  axes.

a)  $\hat{c} = \frac{c}{|c|}$

$$|c| = \sqrt{(3)^2 + (-4)^2 + (-2)^2} = \sqrt{29}$$

$$\therefore \hat{c} = \frac{1}{\sqrt{29}} \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$$

b)  $\cos \theta_x = \frac{x}{|c|} = \frac{3}{\sqrt{29}} \Rightarrow \theta_x = \cos^{-1} \left( \frac{3}{\sqrt{29}} \right) = 56.1^\circ = \text{angle with } x\text{-axis.}$   
 $\cos \theta_y = \frac{y}{|c|} = \frac{-4}{\sqrt{29}} \Rightarrow \theta_y = \cos^{-1} \left( \frac{-4}{\sqrt{29}} \right) = 138^\circ = \text{angle with } y\text{-axis.}$   
 $\cos \theta_z = \frac{z}{|c|} = \frac{-2}{\sqrt{29}} \Rightarrow \theta_z = \cos^{-1} \left( \frac{-2}{\sqrt{29}} \right) = 111.8^\circ = \text{angle with } z\text{-axis.}$

## Solving geometric problems

You need to be able to apply everything from AS and A2 vectors to geometric problems in 3 dimensions. The following facts tend to be useful for such questions:

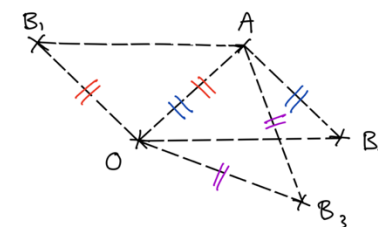
- If we have  $ai + bj + ck = ui + vj + wk$ , then we can compare coefficients on both sides of the equation giving us  $a = u, b = v, c = w$ .
- If the point  $P$  divides the line segment  $AB$  in the ratio  $\lambda : \mu$ , then

$$\vec{OP} = \frac{\lambda}{\lambda + \mu} \vec{OA} + \frac{\mu}{\lambda + \mu} \vec{OB}$$

Drawing a big diagram is always helpful for geometric problems. We will go through an example showcasing the typical style of such questions.

**Example 4:** The points  $A$  and  $B$  have position vectors  $10i-23j+10k$  and  $pi+14j-22k$  respectively, relative to a fixed origin  $O$ , where  $p$  is a constant. Given that  $\triangle OAB$  is isosceles, find three possible positions of the point  $B$ .

We start with a diagram. This helps us to understand why there are three possible positions for  $B$ .



The first possible position is when  $OB$  and  $OA$  are equal, the second when  $OA$  and  $AB$  are equal and the third is when sides  $OB$  and  $AB$  are equal. To find each position vector, we can equate the lengths of the two sides that are equal.

$$B = \begin{pmatrix} p \\ 14 \\ -22 \end{pmatrix}. \text{ So } \vec{AB} = \begin{pmatrix} p \\ 14 \\ -22 \end{pmatrix} - \begin{pmatrix} 10 \\ -23 \\ 10 \end{pmatrix} = \begin{pmatrix} p-10 \\ 37 \\ -32 \end{pmatrix}$$

For  $B_1$ :  $|\vec{OB}| = |\vec{OA}|; \sqrt{(p-10)^2 + 14^2 + (-22)^2} = \sqrt{(10)^2 + (-23)^2 + (10)^2} = 27$   
 $\Rightarrow p^2 + 680 = 27^2 \therefore p = \pm 7$  (squaring both sides and rearranging)

For  $B_2$ :  $|\vec{OA}| = |\vec{AB}|; \sqrt{(10)^2 + (-23)^2 + (10)^2} = \sqrt{(p-10)^2 + (37)^2 + (-32)^2}$   
 $\Rightarrow 27^2 = (p-10)^2 + (37)^2 + (-32)^2$  (squaring both sides and rearranging)  
 $\Rightarrow 729 = p^2 - 20p + 2493$ .

This results in a quadratic where the discriminant is less than 0, so there are no real solutions for  $p$  here.

For  $B_3$ :  $|\vec{OB}| = |\vec{AB}|; \sqrt{(p)^2 + (14)^2 + (-22)^2} = \sqrt{(p-10)^2 + (37)^2 + (-32)^2}$   
 $\Rightarrow p^2 + 680 = (p-10)^2 + 2393$  (squaring both sides and collecting like terms)  
 $\Rightarrow 20p = 1813 \therefore p = \frac{1813}{20}$  (rearranging and finding  $p$ )

So our three possible position vectors of  $B$  are  $\begin{pmatrix} 7 \\ 14 \\ -22 \end{pmatrix}, \begin{pmatrix} -7 \\ 14 \\ -22 \end{pmatrix}$  and  $\begin{pmatrix} 1813 \\ 20 \\ -22 \end{pmatrix}$ .

