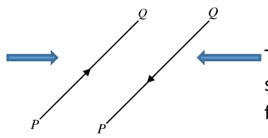
Vectors Cheat Sheet

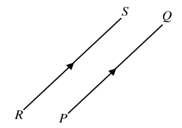
Vectors

A vector has both **magnitude** and **direction.**
You can represent a vector using a **directed line segment.** Direction of arrow shows the direction of vector

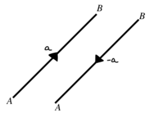
This vector $\overrightarrow{PQ}$ starts at P and finishes at Q



This vector $\overrightarrow{QP}$ starts at Q and finishes at P



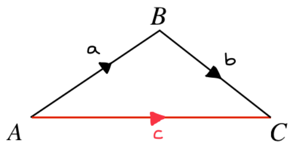
If $\overrightarrow{PQ}=\overrightarrow{RS}$, then the line segment PQ and RS are equal in length and are parallel



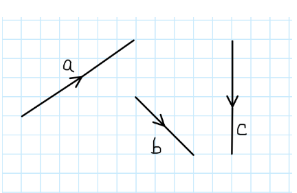
$\overrightarrow{AB}=-\overrightarrow{BA}$ implies
AB is equal in length, parallel and in the opposite direction to BA

Triangle law for vector addition:

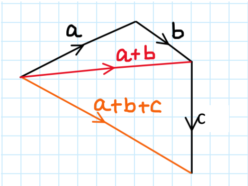
You can add two vector using the triangle law for vector addition.
The law states that
$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$
If $\overrightarrow{AB}$ =**a,** $\overrightarrow{BC}$= **b** and $\overrightarrow{AC}$ = **c** then, **a+b=c**



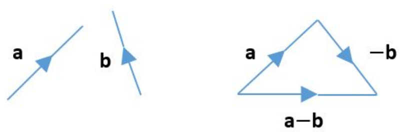
Example 1: The diagram shows vectors a, b and c. Draw a diagram to illustrate the vector addition



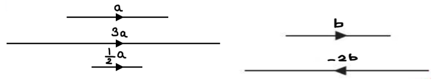
Answer:
First use the triangle law for a+b, then use it again for (a+b) +c.
The resultant goes from the start of a to the end of c



Subtracting a vector is same as adding a negative vector $\boldsymbol{a}-\boldsymbol{b}=\boldsymbol{a}+(-\boldsymbol{b})$

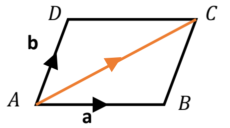


Multiplying a vector:
You can multiply a vector by a scalar. If the number is $+ve$ the resultant vector has same direction but different length. If the number is $-ve$,the new vector has different length but opposite direction



Any vector parallel to the vector **a** may be written as λ**a** where λ is a non-zero scalar

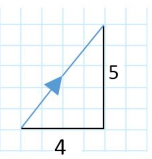
Parallelogram Law for vector addition:
For the parallelogram $ABCD$,
If $\overrightarrow{AB}$ = **a,** $\overrightarrow{AD}$ =**b** then $\overrightarrow{AC}$ = **a+b**



Example 2: Show that the vectors 6**a** + 8**b** and 9**a** + 12**b** are parallel.
Try to write one of the vectors as λ × other vector
9**a** + 12**b** = $\frac{3}{2}$(6**a** + 8**b**)
Since, 9**a** + 12**b** can be written as λ ×(6**a** + 8**b**) where $\lambda=\frac{3}{2}$, this implies that 6**a** + 8**b** and 9**a** + 12**b** are parallel.

Representing vector:

A vector represent change in position or displacement relative to the change in x-and y –axes



$\boldsymbol{a}=\binom{4}{5}$ where 4 is the change in the x –direction and 5 is the change in y-direction
This is called **Column vector.**

To multiply a column vector by a scalar multiply each component by the scalar: $\lambda\binom{p}{q}=\binom{\lambda p}{\lambda q}$

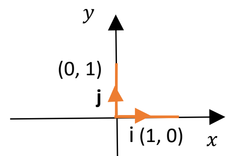
To add 2 column vector add the x- component and the y-components: $\binom{p}{q}+\binom{r}{s}=\binom{p+r}{q+s}$

Example 3: Given that **a**=2**i**+3**j** and **b**=4**i**−**j**, find the vector 4**b**−**a** in terms of **i** and **j**
Start by substituting vectors **a** and **b** in 4**b**−**a** and then simplify
4**b**−**a**= $4(4\mathbf{i}-\mathbf{j})-(2\mathbf{i}+3\mathbf{j})$
$=(16\mathbf{i}-4\mathbf{j})-(2\mathbf{i}+3\mathbf{j})=(16-2)\mathbf{i}+(-4-3)\mathbf{j}=14\mathbf{i}-7\mathbf{j}$
Hence, 4**b**−**a** in terms of **i** and **j** is 14**i** − 7**j**

Example 4: Given that **a**= $\binom{9}{7}$, **b**= $\binom{11}{-3}$ and **c**= $\binom{-8}{-1}$ find 2**a** + **b** + **c**
Start by substituting vectors **a, b** and **c** in 2**a** + **b** + **c**
2**a** + **b** + **c**= $2\binom{9}{7}+\binom{11}{-3}+\binom{-8}{-1}=\binom{18}{14}+\binom{11}{-3}+\binom{-8}{-1}=\binom{21}{10}$

Unit Vector:

A unit vector is a vector of length 1.
The unit vectors along the x- and y-axes are usually denoted by **i** and **j** respectively.
i= $\binom{1}{0}$ **j**= $\binom{0}{1}$



You can write any two-dimensional vector in the form $\binom{p}{q}=p\mathbf{i}+q\mathbf{j}$
In this topic you will learn to find a unit vector in the direction of a given vector
A unit vector in the direction of **a** is $\frac{\mathbf{a}}{|\mathbf{a}|}$
so if $|\mathbf{a}|=5$, then a unit vector in the direction of **a** is $\frac{\mathbf{a}}{5}$

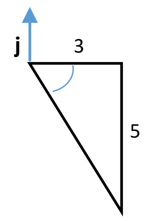
Magnitude and direction:

To calculate the magnitude of vector you can use Pythagoras theorem.
For the vector **a**= $x\mathbf{i}+y\mathbf{j}=\binom{x}{y}$ the magnitude of the vector is given by
$|\mathbf{a}|=\sqrt{x^2+y^2}$
The two straight lines on either side of the vector is the notation for magnitude of a vector

Example 4: **a** = 2**i** + 3**j**, **b** = 3**i** -4**j**. Find the exact value of **a** + **b**
Since we need to calculate exact value of **a** + **b**, first we will have to add the two vectors.
We can do that by representing the vectors in column vector form and then add
So, **a** = $\binom{2}{3}$ and **b** = $\binom{3}{-4}$ implies **a** + **b** = $\binom{2}{3}+\binom{3}{-4}=\binom{2+3}{3-4}=\binom{5}{-1}$
Use the formula for magnitude which is $|\mathbf{a}|=\sqrt{x^2+y^2}$
Hence, $|\mathbf{a}+\mathbf{b}|=\sqrt{5^2+(-1)^2}=\sqrt{26}$
Keep your answer in surd form as you are asked to find exact value.

Example 5: For vector **a** = 5**i** − 12**j**, find the unit vector in the same direction.
A unit vector in the direction of **a** is $\frac{\mathbf{a}}{|\mathbf{a}|}$
Let us first calculate $|\mathbf{a}|$, so $|\mathbf{a}|=\sqrt{5^2+(-12)^2}=\sqrt{25+144}=\sqrt{169}=13$
To find the unit vector in the direction of **a** first write the vector in column vector form and multiply by $\frac{1}{|\mathbf{a}|}$
$\frac{\mathbf{a}}{|\mathbf{a}|}=\frac{\mathbf{a}}{13}=\frac{1}{13}\binom{5}{-12}=\begin{pmatrix}\frac{5}{13}\\ \frac{-12}{13}\end{pmatrix}$
Hence, the unit vector in the direction of **a** is $\begin{pmatrix}\frac{5}{13}\\ \frac{-12}{13}\end{pmatrix}$

Example 6: Find the angle that the vector 3**i** − 5**j** makes with **j** axis

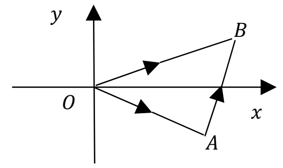


The angle the vector 3**i** − 5**j** makes with **j** is 90° + angle θ
Vector 3**i** − 5**j** on axes will be 3 units to the right on **i** axis and 5 units down on **j** axis
We can find angle θ by using trigonometric ratio $\tan\theta=\frac{opposite}{adjacent}=\frac{5}{3}$
So, $\tan\theta=\frac{5}{3}$ taking inverse we get $\theta=\tan^{-1}\left(\frac{5}{3}\right)=59°$
So the angle 3**i** − 5**j** makes with **j** is 90° + 59° = 149°

Position Vector:

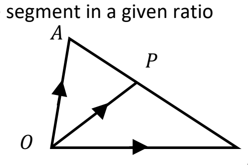
Position vector are the vectors that gives us the position of a point relative to fixed origin
In general, a point P with coordinates (p,q) has a position vector
$\overrightarrow{OP}=p\mathbf{i}+q\mathbf{j}=\binom{p}{q}$

Example7: $\overrightarrow{OA}$ =5**i**−2**j** and $\overrightarrow{AB}$ = 3**i** + 4**j**. Find the position vector of B
Start by drawing the vectors on the axes to find position vector of B. From the diagram position vector of B is vector $\overrightarrow{OB}$.It is easier to work with column vectors, so write the vectors in column vector form
$\overrightarrow{OA}=\binom{5}{-2}$ and $\overrightarrow{AB}=\binom{3}{4}$
Using triangle law of vector addition vector
$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}$
$\overrightarrow{OB}=\binom{5}{-2}+\binom{3}{4}=\binom{5+3}{-2+4}=\binom{8}{2}$
In **i, j** form $\overrightarrow{OB}=8\mathbf{i}+2\mathbf{j}$



Solving geometric problem:

You can use vector to find the position vector of a point that divides a line segment in a given ratio
If the point P divides the line segment AB in the ratio $\lambda:\mu$, then
$\overrightarrow{OP}=\overrightarrow{OA}+\frac{\lambda}{\lambda+\mu}\overrightarrow{AB}$
$=\overrightarrow{OA}+\frac{\lambda}{\lambda+\mu}\overrightarrow{OB}-\overrightarrow{OA}$



Example 8: In the diagram the points A and B have position vectors **a** and **b** respectively.
The point P divides AB in the ratio 1:2. Find the position vector of P
The position vector of P is vector $\overrightarrow{OP}$.You can refer to the figure above. To find $\overrightarrow{OP}$ we can make use of the equation given above for **point P dividing the line segment AB in the ratio $\lambda:\mu$.** Here $\lambda=1$ and $\mu=2$ and substituting them in to the equation gives us,
$\overrightarrow{OP}=\overrightarrow{OA}+\frac{1}{3}\overrightarrow{AB}$
$\overrightarrow{OP}=\overrightarrow{OA}+\frac{1}{3}(\overrightarrow{OB}-\overrightarrow{OA})$

You will have to write $\overrightarrow{AB}$ in terms of the position vectors of A and B in order to find $\overrightarrow{OP}$

$\overrightarrow{OP}=\overrightarrow{OA}+\frac{1}{3}\overrightarrow{OB}-\frac{1}{3}\overrightarrow{OA}$
$=\overrightarrow{OA}-\frac{1}{3}\overrightarrow{OA}+\frac{1}{3}\overrightarrow{OB}=\frac{2}{3}\overrightarrow{OA}+\frac{1}{3}\overrightarrow{OB}=\frac{2}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}$
Hence, $\overrightarrow{OP}$ is $\frac{2}{3}$**a** + $\frac{1}{3}$**b**

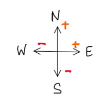
If a and b are two non-parallel vectors and $p\mathbf{a}+q\mathbf{b}=r\mathbf{a}+s\mathbf{b}$ then $p=r$ and $q=s$

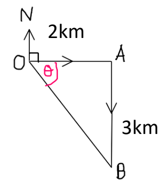
Modelling with vectors

In mechanics, vectors have both magnitude and direction. For example velocity, displacement, and force.
Magnitude of a vector is a scalar quantity - it has size but no direction

Vector quantity	Its magnitude
velocity	speed
displacement	distance

Example 9: A girl walks 2km due east from fixed point O to A, and then 3km due south from A to B. Find: i. The position vector of B relative to O ii. the bearing of B from O
i) To find the position vector of B relative to O,
We will have to represent the girl's journey on a diagram
$\overrightarrow{OB}$ =p**i** + q**i**
2km due east implies p= 2 and 3km due south implies q= −3, you can refer to the diagram here
So $\overrightarrow{OB}$ = (2**i** −3**j**) km





ii) To find bearing of B from O, you will have to calculate angle clockwise from North of O to line segment OB which is 90° + angle θ
Using tan θ we can find angle θ
$\tan\theta=\frac{3}{2}$
$\theta=\tan^{-1}\frac{3}{2}=56.3°$
So the bearing of B from O is 90° + 56.3° = 146.3° = 146°

Example 10: Find the distance moved by particle which travels for 10 seconds at velocity (5**i**−**j**) ms^{-1}
You are familiar with the formula for distance, Distance = speed × time
In order to calculate distance we need to find speed (i.e. magnitude of velocity)
and substitute speed and time = 10 seconds in the formula.
Use the magnitude formula to calculate speed
Speed = $|5\mathbf{i}-\mathbf{j}|=\sqrt{5^2+(-1)^2}=\sqrt{25+1}=\sqrt{26}=5.099$
Hence, Distance = 5.099× 10 = 51.0cm