1. Using appropriate right-angled triangles, show that tan 45° = 1 and $\tan 30° = \frac{1}{\sqrt{3}}$.

Hence show that $\tan 75° = 2 + \sqrt{3}$.

[7]

2. You are given that f(x) = cos x + λ sin x where λ is a positive constant.

i. Express f(x) in the form R cos (x – α), where R > 0 and $0 < \alpha < \frac{1}{2}\pi$, giving R and α in terms of λ.

[4]

ii. Given that the maximum value (as x varies) of f(x) is 2, find R, λ and α, giving your answers in exact form.

[4]

3. Express cos θ – 3 sin θ in the form R cos(θ + α), where R > 0 and $0 < \alpha < \frac{\pi}{2}$.

Hence show that the equation cos θ – 3 sin θ = 4 has no solution.

[6]

4. In Fig. 8, OAB is a thin bent rod, with OA = 1 m, AB = 2 m and angle OAB = 120°. Angles θ, ϕ and h are as shown in Fig. 8.

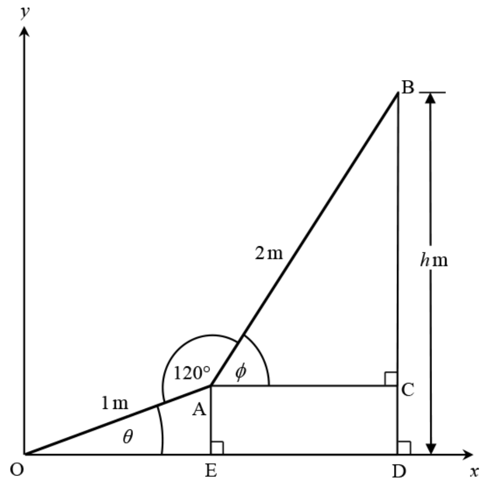


Fig. 8

(a) Show that $h = \sin\theta + 2\sin(\theta + 60°)$. [3]

The rod is free to rotate about the origin so that θ and ϕ vary. You may assume that the result for h in part (a) holds for all values of θ.

(b) Find an angle θ for which h = 0. [5]

5. (a) Express $\cos\theta + 2\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $0 < \alpha < \frac{1}{2}\pi$ and R is positive and given in exact form. [4]

The function $f(\theta)$ is defined by $f(\theta) = \dfrac{1}{(k + \cos\theta + 2\sin\theta)}$, $0 \le \theta \le 2\pi$, k is a constant.

(b) The maximum value of $f(\theta)$ is $\dfrac{(3+\sqrt{5})}{4}$. Find the value of k. [3]

6. (See Insert for Specimen 64003.) Fig. 15 shows a unit circle and the escribed regular polygon with 12 edges.

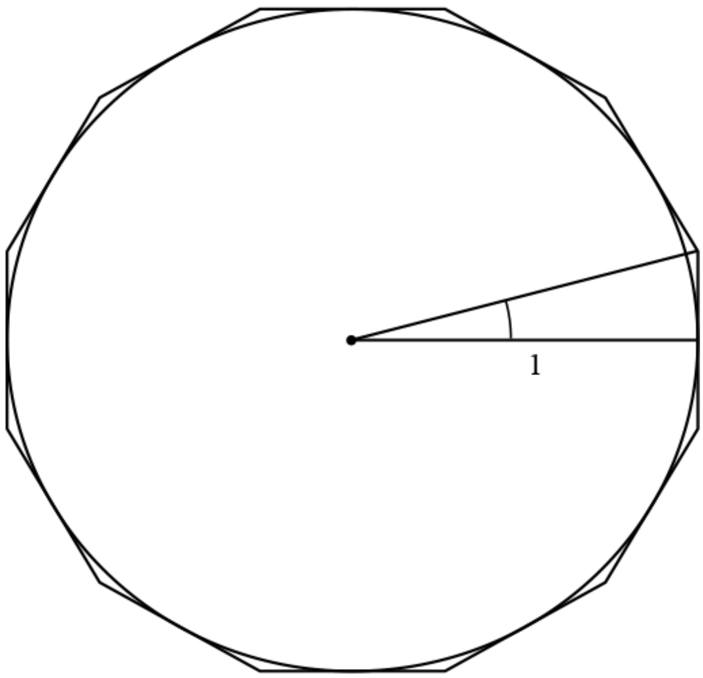


Fig. 15

(a) Show that the perimeter of the polygon is 24 tan15° . [2]

(b) Using the formula for tan $(\theta - \phi)$ show that the perimeter of the polygon is $48 - 24\sqrt{3}$. [3]

7. (a) Express 2 cos θ + 3 sin θ in the form R sin ($\theta + \alpha$), where $0 < \alpha < \frac{1}{2}\pi$ and R is a positive constant given in exact form. [4]

(b) Determine the set of values of k for which the curve $y = k + 2 \cos x + 3 \sin x$ lies completely above the x-axis. [4]

(c) Explain why the curve $y = \dfrac{1}{k + 2\cos x + 3\sin x}$ lies completely above the x-axis for the set of values of k found in part **(b).** [1]

8. (a) Write down the exact values of tan 45° and tan 60°. [1]

(b) **In this question you must show detailed reasoning.**

Show that $\tan 15° = 2 - \sqrt{3}$. [4]

9. **In this question you must show detailed reasoning.**

(a) Express 8 cos x + 5 sin$x$ in the form R cos(x – α), where R and α are constants with $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. [3]

(b) Hence solve the equation 8 cos$x$ + 5 sin$x$ = 6 for $0 \le x < 2\pi$, giving your answers correct to 4 decimal places. [3]

10. (a) Express 7cos$x$ – 24sin$x$ in the form R cos($x + \alpha$), where $0 < \alpha < \dfrac{\pi}{2}$. [3]

(b) Write down the range of the function

$$f(x) = 12 + 7\cos x - 24\sin x, \qquad 0 \le x \le 2\pi.$$

[2]

11. (a) Express $\sqrt{2}\cos x - \sin x$ in the form $R\cos(x + \alpha)$, where $0 < \alpha < \frac{\pi}{2}$. [3]

(b) You are given that

$$f(x) = \frac{5}{2 + \sqrt{2}\cos x - \sin x} \text{ for } 0 \leq \alpha \leq 2\pi.$$

Find the minimum value of f(x), giving your answer in the form $a + b\sqrt{c}$ where a, b and c are integers to be determined. [3]

12. (a) Write $\cos^2 x$ in terms of $\cos 2x$. [1]

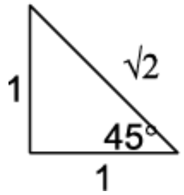
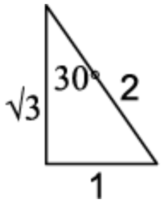
(b) Express $6 \sin 2x + 8 \cos 2x$ in the form $R\cos(2x - \theta)$, where $0 < \theta < \frac{\pi}{2}$. [2]

In this question you must show detailed reasoning.

(c) Hence solve the equation $6 \sin 2x + 16 \cos^2 x = 13$ for $0 \leq x \leq 2\pi$ giving your answers correct to 3 significant figures. [5]

END OF QUESTION paper

Mark scheme

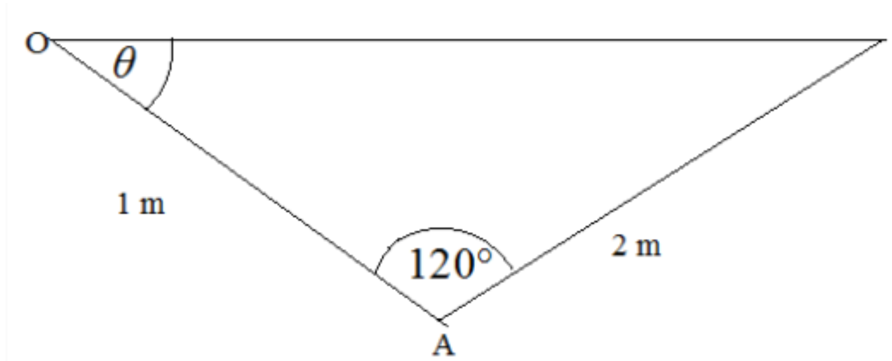
Question	Answer/Indicative content	Marks	Guidance
1	 $\tan 45° = 1/1 = 1$*  $\tan 30° = 1/\sqrt{3}$*		For both B marks **AG** so need to be convinced and need triangles but further explanation need not be on their diagram. Any given lengths must be consistent.
		B1	Need √2 or indication that triangle is isosceles oe
		B1	Need all three sides oe
	tan 75° = tan (45° + 30°)	M1	use of **correct** compound angle formula with 45°, 30° soi
	$= \dfrac{\tan 45 + \tan 30}{1 - \tan 45 \tan 30} = \dfrac{1 + 1/\sqrt{3}}{1 - 1/\sqrt{3}}$	A1	substitution in terms of √3 in any **correct** form
	$= \dfrac{1+\sqrt{3}}{-1+\sqrt{3}}$ $= \dfrac{(1+\sqrt{3})^2}{3-1}$	M1	eliminating fractions within a fraction (or rationalising, whichever comes first) provided compound angle formula is used as tan(*A* + *B*) = tan(*A* ± *B*)/(1 ± tan*A*tan*B*).
	(oe eg $\dfrac{3+\sqrt{3}}{3-\sqrt{3}} = \dfrac{(3+\sqrt{3})^2}{9-3}$)	M1	rationalising denominator (or eliminating fractions whichever comes second)
	$= \dfrac{(3+2\sqrt{3}+1)}{3-1} = 2+\sqrt{3}$ *	A1	**correct** only, **AG** so need to see working

					Examiner's Comments There were some good explanations with appropriate triangles in the first part. However, too many candidates felt it was enough to only give the information given in the question and this was not sufficient. More was needed than, for example, a right-angled triangle with lengths of 1, 1 and 45° to show that tan 45°=1. It was necessary to clearly show the triangle was isosceles by giving the other angle or showing that the hypotenuse was √2, or equivalent. Some made errors when calculating the other lengths in both triangles. Some good candidates failed to score here seemingly being unfamiliar with where these identities came from. The second part started well for most candidates, who usually used the correct compound angle formula, (although there were a few who thought that tan75°=tan45°+tan30°) and made the first substitution. Thereafter, this question gave the opportunity for candidates to show that they could eliminate fractions within fractions and rationalise the denominator. This was a good discriminator for the higher scoring candidates. A few candidates abandoned their attempt at half way and equated $\dfrac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}$ at that stage to the given answer 2+√3.
			Total	7	
2		i	cos x + λsin x = R cos(x – α)	Enter text here.	Enter text here.
		i	= R cos x cos α + R sin x sin α		Enter text here.
		i	⇒ R cos α = 1, R sin α = λ	M1	Correct pairs. Condone sign error (so accept R sin α = – λ)
		i	⇒ R^2 = 1 + λ², R = √(1 + λ²)	B1	Positive square root only – isw. Accept R = 1/cos(arctan λ)or R = λ/sin(arctan λ)
		i	tan α = λ (oe)	M1	Follow through their pairs. tan α = λ with no working implies both M marks. However, cos α = 1, sin α = λ ⇒ tan α = λ scores M0M1. First two M marks may be implied by combining one of the pairs with R

		i	⇒ α = arctan λ (oe)	A1	eg, $\cos\alpha = \frac{1}{\sqrt{(1+\lambda^2)}}$ or $\sin\alpha = \frac{\lambda}{\sqrt{(1+\lambda^2)}}$ $\alpha = \arccos\left(\frac{1}{\sqrt{(1+\lambda^2)}}\right)$, $\alpha = \arcsin\left(\frac{\lambda}{\sqrt{(1+\lambda^2)}}\right)$ Accept embedded answers, eg, $\sqrt{(1 + \lambda^2)}\cos(x - \lambda)$ for full marks
		ii	max is R so $R = 2$	B1	Enter text here.
		ii	$1 + \lambda^2 = 4 \Rightarrow \lambda = \sqrt{3}$	M1 A1	M1 for using $\sqrt{(1 + \lambda^2)} = R_{max}$, A0 for $\pm\sqrt{3}$ as final answer
		ii	$\alpha = \arctan\sqrt{3} = \pi/3$	B1	www (eg $\lambda = 1$ and $\cos\alpha = (1 + \lambda)^{-1} \Rightarrow \alpha = \pi/3$ is B0) Exact answers only for final A and B marks **Examiner's Comments** This question differentiated well due to the coefficient of sin x taking the form of a positive constant rather than a number. Many candidates, however, were unfazed by this and worked out the correct values for R and α . Some candidates lost the first method mark by not including R inthe expanded trigonometric statements $R\cos\alpha = 1$, $R\sin\alpha = \lambda$. Writing α in terms of the more complex arcsin and arccos expressions was surprisingly common. It was a littleworrying that a sizeable minority of candidates went from the correct $R = \sqrt{1+\lambda^2}$ to the incorrect $R = 1 + \lambda$, thinking the squared terms and the square root cancelled each other out. In part (ii) those candidates that realised that $R = 2$ usually went on to get the correct values for λ and α. However it was common for λ to be incorrect due to an incorrect expression for R from part (i). A fair proportion of candidates gave α in degrees and those who gave α as either arccos $\left(\frac{1}{\sqrt{1+\lambda^2}}\right)$ or arcsin $\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)$

					were generally less successful in this part than those who gave α as arctanλ.
			Total	8	
3			$\cos\theta - 3\sin\theta = R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$ $\Rightarrow 1 = R\cos\alpha, 3 = R\sin\alpha$	M1A1	Correct pairs. Condone sign errors for the M mark (so accept $R\sin\alpha = -3$)
			$R^2 = 1^2 + 3^2 = 10 \Rightarrow R = \sqrt{10}$	B1	Or 3.2 or better, not $\pm\sqrt{10}$ unless $+\sqrt{10}$ chosen
			$\tan\alpha = 3 \Rightarrow \alpha = 1.249$	M1 A1	ft their pairs (condone sign errors but division must be the correct way round),A1 for 1.249 or better (accept 1.25), with no errors seen in method for angle
			Maximum value of $\cos\theta - 3\sin\theta$ is $\sqrt{10} < 4$	B1	Or equivalent convincing numerical statement that no solutions exist e.g. $\frac{4}{\sqrt{10}} > 1$. Maybe embedded in an attempt at a solution. Do not accept general statements e.g. 'doesn't work' – must be clear why no solutions exist – **dependent on first B1** **SC**: If candidates state that $\cos\alpha = 1, \sin\alpha = 3 \Rightarrow \tan\alpha = 3$ this could score M0A0B1M1A1B1 (so max 4/6) **Note that those candidates who state** $R = \sqrt{10}$ **and** $\tan\alpha = 3$ **with no (wrong) working seen could go on to score full marks** **Examiner's Comments** The majority of candidates correctly worked out the values of R and α although some candidates lost the first method mark by not including R in the expanded trigonometric statements $R\cos\alpha = 1$ and $R\sin\alpha = 3$. Some failed to give α in radians and a small minority stated R as 10 rather than the correct $\sqrt{10}$. Candidates were less successfully in showing that $\cos\theta - 3\sin\theta = 4$ had no solutions with many simply stating that $\theta + 1.249 = \arccos\left(\frac{4}{\sqrt{10}}\right)$

					‘does not work’ or gives a ‘math error’. Many candidates failed to explain or give an equivalent mathematical statement that the maximumvalue of cos θ – 3 sin θ is $\sqrt{10}$ which is less than 4 and so did not score the final mark in this question.
			Total	6	
4		a	<table><tr><td></td><td>BAC = 360 – 120 – 90 – (90 – θ)</td></tr><tr><td></td><td>= θ + 60</td></tr><tr><td>⇒</td><td>BC = 2 sin(θ + 60)</td></tr><tr><td></td><td>CD = AE = sin θ</td></tr></table>⇒ h = CD + BC = sin θ +2 sin (θ + 60°)	B1(AO3.1a) M1(AO1.1) E1(AO2.1) [3]	AG
		b	<table><tr><td></td><td>h = sin θ + 2sin (θ + 60°)</td></tr><tr><td></td><td>= sin θ + 2(sinθ cos 60 + cosθ sin 60)</td></tr><tr><td></td><td>= sin θ + sin θ + $\sqrt{3}$cos θ</td></tr><tr><td></td><td>= 2sin θ + $\sqrt{3}$cos θ</td></tr><tr><td>h = 0 ⇒</td><td>2sin θ + $\sqrt{3}$ cos θ = 0</td></tr><tr><td>⇒</td><td>$\tan\theta = -\frac{\sqrt{3}}{2}$</td></tr><tr><td>⇒</td><td>θ = –40.9° [so 40.9° below the horizontal]</td></tr></table>**Alternative method** Diagram with h = 0	M1(AO3.1a) A1(AO2.1) M1(AO1.1) M1(AO1.1) A1(AO1.1) M1(AO3.1a) M1(AO2.1) A1(AO1.1)	use of compound angle formula h = 0 soi Use of $\frac{\sin}{\cos} = \tan$ or 319.1° or 139.1°

			 $a^2 = 1^2 + 2^2 - 4\cos 120°$ $a = \sqrt{7}$ $\sin\theta = \dfrac{2\sin 120°}{\sqrt{7}} = \sqrt{\dfrac{3}{7}}$ $\theta = -40.9°$[so 40.9° below the horizontal]	M1(AO1.1) A1(AO1.1) [5]	For final mark, θ shown below horizontal in diagram together with 40.9° is accepteble
			Total	8	
5		a	$\cos\theta + 2\sin\theta°\ R\cos(\theta - \alpha)$ $\Rightarrow R\cos\alpha = 1,\ R\sin\alpha = 2$ $\Rightarrow R2 = 5,\ R = \sqrt{5}$ $\tan\alpha = 2,\ \alpha = 1.107$	M1(AO1.1a) B1(AO1.1) M1A1(AO1.1 1.1) [4]	
		b	max value is $\dfrac{1}{(k - \sqrt{5})}$	M1(AO3.1a) M1(AO1.1)	

			$\frac{1}{\left(k - \sqrt{5}\right)} = \frac{\left(3+\sqrt{5}\right)}{4}$ $4 = 3k - 5 + k\sqrt{5} - 3\sqrt{5}$ [This is indep of $\sqrt{5}$ so] $k = 3$	A1(AO1.1) [3]	
			Total	7	
6		a	Angle = 360 ÷ 24 = 15 Edge length = 2 tan15° Perimeter = 12 × 2 tan15° = 24 tan15°	M1(AO1.1) E1(AO2.1) [2]	AG
		b	tan15° = tan (45° – 30°) $= \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} \quad \left[= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\left(\sqrt{3}-1\right)^2}{2}\right]$ **Alternative method** tan15° = tan (60° – 45°) $= \frac{\sqrt{3}-1}{1+\sqrt{3}} \quad \left[= \frac{2\sqrt{3}-4}{-2}\right]$	B1(AO3.1a) M1(AO1.1) B1(AO3.1a) M1(AO1.1) E1(AO2.1) [3]	Exact values of tan 45° and tan30° used Exact values of tan 60° and tan15° used

			Answer	Marks	Guidance	
			Perimeter = 12 × 2 tan15° = $48-24\sqrt{3}$		Correct completion AG	
			Total	5		
7		a	$2\cos\theta + 3\sin\theta \equiv R\sin(\theta + \alpha) \Rightarrow R\cos\alpha = 3,\ R\sin\alpha = 2$ So $R^2 = 13 \Rightarrow R = \sqrt{13}$ and $\tan\alpha = \frac{2}{3}$ $\Rightarrow \alpha = 0.588$	M1(AO 1.1a) B1(AO 1.1) M1(AO 1.1) A1(AO 1.1) [4]		
		b	$k + 2\cos x + 3\sin x > 0$ [for all x] $\sqrt{13}\sin(x+0.588)+k>0$ Minimum value of LHS is $k-\sqrt{13}$ $k>\sqrt{13}$	B1(AO 3.1a) M1(AO 1.1) M1(AO 3.1a) A1(AO 2.2a) [4]	oe Use of expression from part **(a)** Attempt to find minimum value	May be by calculus

		c	$k+2\cos x+3\sin x>0 \Rightarrow \dfrac{1}{k+2\cos x+3\sin x}>0$	E1(AO 2.4) [1]	oe; accept e.g. statement that the reciprocal of a positive number is positive	
			Total	9		
8		a	$\tan 45^0 = 1$ and $\tan 60^0 = \surd 3$	B1(AO 1.2) [1]		
		b	$\tan(60-45)=\dfrac{\tan 60-\tan 45}{1+\tan 60\times\tan 45}$ $\dfrac{\sqrt{3}-1}{1+\sqrt{3}}$ Multiply numerator and denominator by $\surd 3-1$ eg $\dfrac{3-2\sqrt{3}+1}{3-1}$ $=2-\surd 3$	M1(AO 3.1a) M1(AO 1.1) M1(AO 1.1) A1(AO 2.1) [4]	**DR** Substitution of their surds in correct compound angle formula **AG** Convincing arithmetic to given result	Other correct methods eg use of double angle formula are acceptable
			Total	5		

9		a	DR 8 cos x + 5 sin x = R(cos x cos α + sin x sin α), so 8 = R cos α and 5 = R sin α $R=\sqrt{8^2+5^2}=\sqrt{89}$ $\alpha=\arctan\left(\frac{5}{8}\right)$ $8\cos x+5\sin x=\sqrt{89}\cos\left(x-\arctan\left(\frac{5}{8}\right)\right)$	M1(AO1.1a) B1(AO1.1b) A1(AO1.1b) [3]	Equating coefficients Accept 9.43 or better Accept 0.559 or better (No penalty for omission of this step)
		b	DR $\cos\left(x-\arctan\left(\frac{5}{8}\right)\right)=\frac{6}{\sqrt{89}}$, so $x-\arctan\left(\frac{5}{8}\right)=0.881\,49\ldots$ or $2\pi-0.881\,49\ldots$ x = 1.4401 x = 5.9603	M1(AO1.1a) A1(AO1.1a) A1(AO1.1a) [3]	Method leading to at least one solution If a rounded value from (a) used max. A1 only
			Total	6	
10		a	R = 25	B1 (AO 1.1)	

			$\tan^{-1}\left(\frac{24}{7}\right)$ or $\sin^{-1}\left(\frac{24}{25}\right)$ or $\cos^{-1}\left(\frac{7}{25}\right)$ $25\cos(x + 1.29)$	M1 (AO 1.1) A1 (AO 1.1) [3]	$\alpha = 1.28700221759$ rounded to 2 or more sf | 73.739795° rounded to 2 or more sf may imply **M1A0** allow **A1** for α found to 2 or more sf Examiner's Comments The majority of candidates gained full credit, with careless arithmetic resulting in dropped accuracy marks on this routine item.
		b	12 ± *their* 25 $-13 \le f(x) \le 37$	M1 (AO 3.1a) A1 (AO 1.1) [2]	or one of – 13 and 37 identified allow eg from – 13 to 37 inclusive | **A0** if inequality is strict Examiner's Comments Some candidates answered their own question, taken directly from part (a).
			Total	5	
11		a	$R = \sqrt{3}$ $\tan^{-1}(\frac{1}{\sqrt{2}})$	B1 (AO1.1) M1 (AO1.1) A1 (AO1.1) [3]	

			α = 0.615		0.61547970… rounded to 2 or more significant figures
		b	$f(x) = \dfrac{5}{2+\sqrt{3}\cos(x+0.62)}$ At min value, $\cos(x + 0.62) = 1$ soi $10 - 5\sqrt{3}$	M1 (AO3.1a) M1 (AO2.1) A1 (AO1.1) [3]	FT their R BC rationalising
			Total	6	
12		a	[$\cos^2 x$ =] ½(1 + cos2x)	B1 (AO 1.1a) [1]	
		b	R = 10 θ = arctan(0.75) isw or 0.643501… to 3 or more sf	B1 (AO 1.1) B1 (AO 1.1) [2]	
		c	DR substitution of results from parts (a) and (b) in the equation 6sin2x + 8cos2x = 5 $\arccos(\frac{5}{R})$ found FT their R 0.845, 3.99, 2.94, 6.08 cao	M1 (AO 2.1) A1 (AO 1.1) M1 (AO 3.1a) A1 (AO 1.1) A1 (AO 1.1)	

				[5]	if **A0A0** allow **A1** for all four values correct to a different precision
			Total	8	