1. (a) Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$

Give the exact value of R and the value of α in radians to 3 decimal places.

(3)

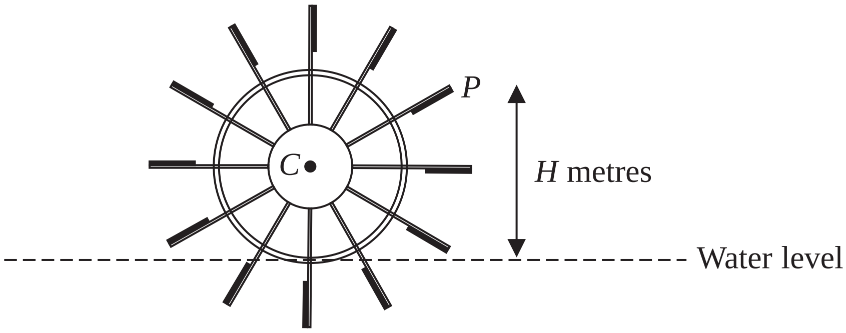


Figure 6

Figure 6 shows the cross-section of a water wheel.

The wheel is free to rotate about a fixed axis through the point C.

The point P is at the end of one of the paddles of the wheel, as shown in Figure 6.

The water level is assumed to be horizontal and of constant height.

The vertical height, H metres, of P above the water level is modelled by the equation

$$H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$$

where t is the time in seconds after the wheel starts rotating.

Using the model, find

(b) (i) the maximum height of P above the water level,

(ii) the value of t when this maximum height first occurs, giving your answer to one decimal place.

(3)

In a single revolution of the wheel, P is below the water level for a total of T seconds.

According to the model,

(c) find the value of T giving your answer to 3 significant figures.

(*Solutions based entirely on calculator technology are not acceptable*.)

(4)

In reality, the water level may not be of constant height.

(d) Explain how the equation of the model should be refined to take this into account.

(1)

a) $2\cos\theta - \sin\theta = R\cos(\theta+\alpha)$
$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

comparing coefficients:
$R\cos\alpha = 2$
$R\sin\alpha = 1$

$R = \sqrt{1^2+2^2} = \sqrt{5}$ 1

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{2} \Rightarrow \tan\alpha = \frac{1}{2} \Rightarrow \alpha = 0.464$ (3d.p) 1 1

b) $H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$

(i) max H

$H = 3 + 2(2\cos(0.5t) - \sin(0.5t))$
$= 3 + 2\sqrt{5}\cos(0.5t + 0.464)$

max H happens when $\cos(0.5t + 0.464) = 1$, i.e. $H = 3 + 2\sqrt{5}$ 1

(ii) $\Rightarrow$ $0.5t + 0.464 = \cos^{-1}(1)$ 1
$= 2\pi$
$0.5t = 2\pi - 0.464$
$t = 11.6$ (1dp) 1

c) $H = 3 + 2\sqrt{5}\cos(0.5t + 0.464)$

T seconds when $H < 0$. Find T.

$$3 + 2\sqrt{5}\cos(0.5t + 0.464) = 0 \quad 1$$

Method: find two consecutive times when $t = 0$, then find the difference between them

$$\cos(0.5t + 0.464) = -\frac{3}{2\sqrt{5}}$$

$$0.5t + 0.464 = \cos^{-1}\left(-\frac{3}{2\sqrt{5}}\right)$$

$$t = 2\left(\cos^{-1}\left(-\frac{3}{2\sqrt{5}}\right) - 0.464\right) \quad 1$$

1

time required is e.g. $2(3.977... - 0.464) - 2(2.306.. - 0.464)$

$= 3.34$ (3sf) 1

d) the 3 would need to be variable. 1

2.

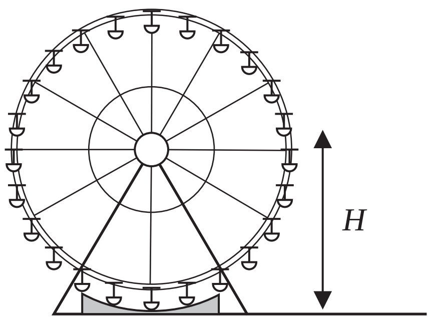


Figure 4

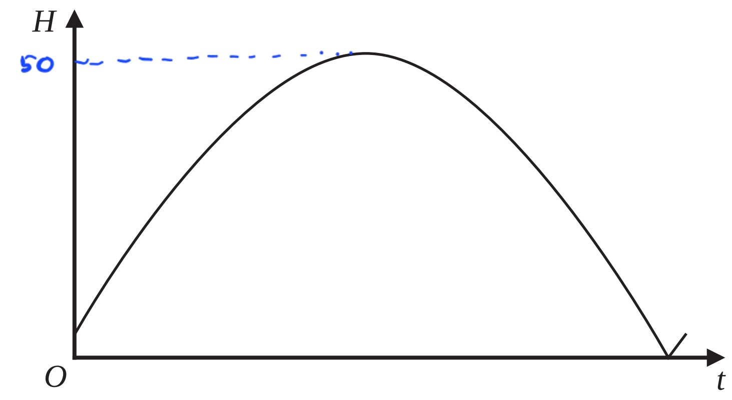


Figure 5

Figure 4 shows a sketch of a Ferris wheel.

The height above the ground, H m, of a passenger on the Ferris wheel, t seconds after the wheel starts turning, is modelled by the equation

$$H = |A\sin(bt + \alpha)°|$$

where A, b and α are constants.

Figure 5 shows a sketch of the graph of H against t, for one revolution of the wheel.

Given that

- the maximum height of the passenger above the ground is 50 m
- the passenger is 1 m above the ground when the wheel starts turning
- the wheel takes 720 seconds to complete one revolution

(a) find a complete equation for the model, giving the exact value of A, the exact value of b and the value of α to 3 significant figures.

(4)

(b) Explain why an equation of the form

$$H = |A\sin(bt + \alpha)°| + d$$

where d is a positive constant, would be a more appropriate model.

(1)

a) $H = |A \sin(bt + \alpha)|$

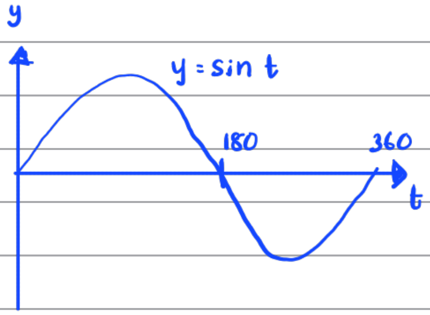
because max $\sin(bt + a) = 1$

maximum $H = A = 50$ 1

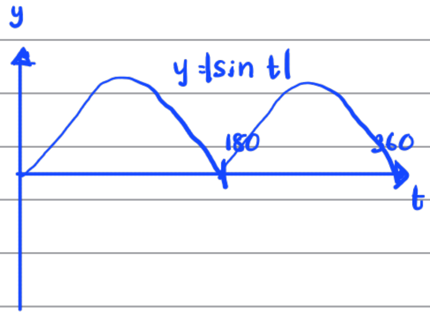
when $t = 0$, $H = 1 = 50 \sin \alpha$

$\frac{1}{50} = \sin \alpha$

$\alpha = 1.15$ 1



period = 360



period = 180

$$|\sin t| = 180^\circ$$

$$|\sin bt| = \frac{180^\circ}{b} = 720^\circ$$

$$= 180^\circ = 720^\circ b$$

$$b = \frac{1}{4} \quad (1)$$

$$H = \left| 50 \sin\left(\frac{1}{4}t + 1.15\right) \right| \quad (1)$$

b) We want the lowest point of the wheel to be a distance of d above ground, so that the passenger does not touch the ground. Hence, a model of the form of $H = |A \sin(bt + \alpha)| + d$ would be more appropriate. (1)

3. On a roller coaster ride, passengers travel in carriages around a track.

On the ride, carriages complete multiple circuits of the track such that

- the maximum vertical height of a carriage above the ground is 60 m
- a carriage starts a circuit at a vertical height of 2 m above the ground
- the ground is horizontal

The vertical height, H m, of a carriage above the ground, t seconds after the carriage starts the first circuit, is modelled by the equation

$$H = a - b(t - 20)^2$$

where a and b are positive constants.

(a) Find a complete equation for the model.

(3)

(b) Use the model to determine the height of the carriage above the ground when $t = 40$

(1)

In an alternative model, the vertical height, H m, of a carriage above the ground, t seconds after the carriage starts the first circuit, is given by

$$H = 29\cos(9t + \alpha)° + \beta \qquad 0 \leqslant \alpha < 360°$$

where α and β are constants.

(c) Find a complete equation for the alternative model.

(2)

Given that the carriage moves continuously for 2 minutes,

(d) give a reason why the alternative model would be more appropriate.

(1)

a) $H = a - b(t-20)^2$

given:
1. turning point is when $H = 60$
2. when $t=0$, $H=2$

turning point has $H=60$ and $t=20$ $\therefore$ $60 = a + b(20-20)$
$a = 60$ (1)

sub in $t=0$, $h=2$:

(1)

$2 = 60 - b(0-20)^2 \Rightarrow 2 = 60 - 400b$

$400b = 58$

$b = 0.145$

$H = 60 - 0.145(t-20)^2$ (1)

b) sub in $t = 40$

$H = 60 - 0.145(40-20)^2$

$= 2\text{ m}$ (1)

c) $H = 29\cos(9t + \alpha) + \beta$

$\frac{dH}{dt} = -261\sin(9t + \alpha) = 0$ when $t = 20$

$\therefore \sin(180 + \alpha) = 0$

$180 + \alpha = 0 \Rightarrow \alpha = -180°$, out of range

$180 + \alpha = 360 \Rightarrow \alpha = 180°$, in range (1)

$H = 29\cos(9t + 180) + \beta$ $(0 \leq \alpha < 360)$

sub in $t = 0$, $H = 2$

$2 = 29\cos 180 + \beta$

$2 = -29 + \beta$

$\beta = 31$

$H = 29\cos(9t + 180) + 31$ (1)

d) The alternative model allows for more than one circuit (1)