Questions

Q1.

(a) Express $10 \cos \theta - 3 \sin \theta$ in the form $R \cos (\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$

Give the exact value of R and give the value of α, in degrees, to 2 decimal places.

(3)

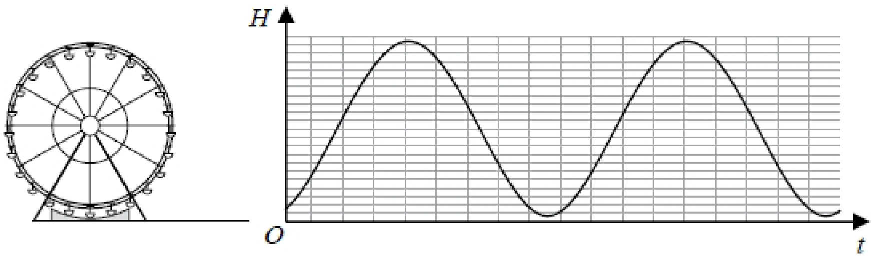


Figure 3

The height above the ground, H metres, of a passenger on a Ferris wheel t minutes after the wheel starts turning, is modelled by the equation

$$H = a - 10 \cos (80\,t)° + 3 \sin (80\,t)°$$

where a is a constant.

Figure 3 shows the graph of H against t for two complete cycles of the wheel.

Given that the initial height of the passenger above the ground is 1 metre,

(b) (i) find a complete equation for the model,

(ii) hence find the maximum height of the passenger above the ground.

(2)

(c) Find the time taken, to the nearest second, for the passenger to reach the maximum height on the second cycle.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(3)

It is decided that, to increase profits, the speed of the wheel is to be increased.

(d) How would you adapt the equation of the model to reflect this increase in speed?

(1)

(Total for question = 9 marks)

Q2.

The depth of water, *D* metres, in a harbour on a particular day is modelled by the formula

$$D = 5 + 2\sin(30t)^\circ \qquad 0 \leqslant t < 24$$

where *t* is the number of hours after midnight.

A boat enters the harbour at 6:30 am and it takes 2 hours to load its cargo.
The boat requires the depth of water to be at least 3.8 metres before it can leave the harbour.

(a) Find the depth of the water in the harbour when the boat enters the harbour.

(1)

(b) Find, to the nearest minute, the earliest time the boat can leave the harbour.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(4)

(Total for question = 5 marks)

Q3.

$$f(x) = 10e^{-0.25x}\sin x, \quad x \geqslant 0$$

(a) Show that the x coordinates of the turning points of the curve with equation $y = f(x)$ satisfy the equation $\tan x = 4$

(4)

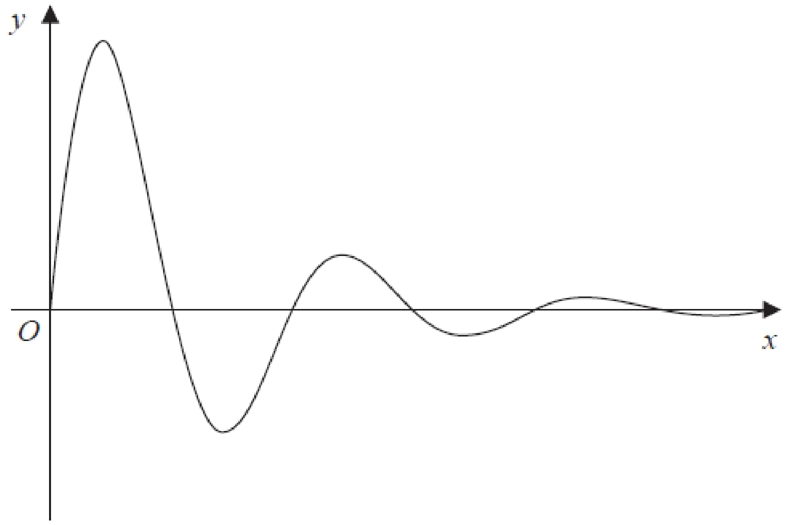


Figure 3

Figure 3 shows a sketch of part of the curve with equation $y = f(x)$.

(b) Sketch the graph of H against t where

$$H(t) = \left|10e^{-0.25t}\sin t\right| \quad t \geqslant 0$$

showing the long-term behaviour of this curve.

(2)

The function H(t) is used to model the height, in metres, of a ball above the ground t seconds after it has been kicked.

Using this model, find

(c) the maximum height of the ball above the ground between the first and second bounce.

(3)

(d) Explain why this model should not be used to predict the time of each bounce.

(1)

(Total for question = 10 marks)

Q4.

(a) Express $\sin x + 2\cos x$ in the form $R\sin(x + \alpha)$ where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α in radians to 3 decimal places.

(3)

The temperature, θ °C , inside a room on a given day is modelled by the equation

$$\theta = 5 + \sin\left(\frac{\pi t}{12} - 3\right) + 2\cos\left(\frac{\pi t}{12} - 3\right) \qquad 0 \leqslant t < 24$$

where t is the number of hours after midnight.

Using the equation of the model and your answer to part (a),

(b) deduce the maximum temperature of the room during this day,

(1)

(c) find the time of day when the maximum temperature occurs, giving your answer to the nearest minute.

(3)

(Total for question = 7 marks)

Q5.

(a) Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and the value of α in radians to 3 decimal places.

(3)

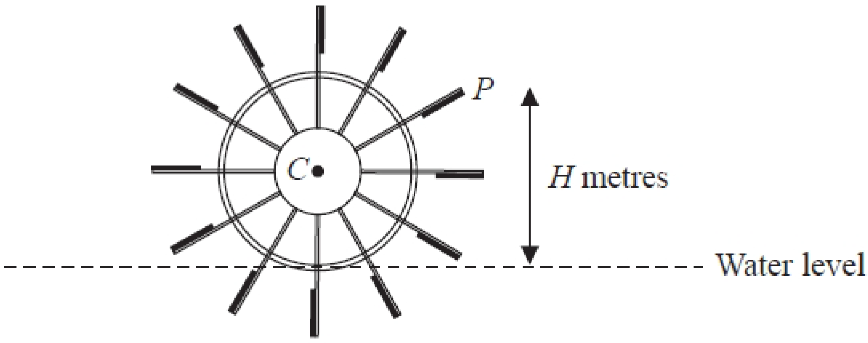


Figure 6

Figure 6 shows the cross-section of a water wheel.

The wheel is free to rotate about a fixed axis through the point C.

The point P is at the end of one of the paddles of the wheel, as shown in Figure 6.

The water level is assumed to be horizontal and of constant height.

The vertical height, H metres, of P above the water level is modelled by the equation

$$H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$$

where t is the time in seconds after the wheel starts rotating.

Using the model, find

(b) (i) the maximum height of P above the water level,

(ii) the value of t when this maximum height first occurs, giving your answer to one decimal place.

(3)

In a single revolution of the wheel, P is below the water level for a total of T seconds.

According to the model,

(c) find the value of T giving your answer to 3 significant figures.

(Solutions based entirely on calculator technology are not acceptable.)

(4)

In reality, the water level may not be of constant height.

(d) Explain how the equation of the model should be refined to take this into account.

(1)

(Total for question = 11 marks)

Mark Scheme

Q1.

Question	Scheme	Marks	AOs
(a)	$R=\sqrt{109}$	B1	1.1b
	$\tan\alpha=\frac{3}{10}$	M1	1.1b
	$\alpha=16.70°$ so $\sqrt{109}\cos(\theta+16.70°)$	A1	1.1b
		(3)	
(b)	(i) e.g $H=11-10\cos(80t)°+3\sin(80t)°$ or $H=11-\sqrt{109}\cos(80t+16.70)°$	B1	3.3
	(ii) $11+\sqrt{109}$ or 21.44 m	B1ft	3.4
		(2)	
(c)	Sets $80t+"16.70"=540$	M1	3.4
	$t=\frac{540-"16.70"}{80}=(6.54)$	M1	1.1b
	$t=6$ mins 32 seconds	A1	1.1b
		(3)	
(d)	Increase the '80' in the formula For example use $H=11-10\cos(90t)°+3\sin(90t)°$		3.3
		(1)	
			(9 marks)

Notes:

(a)
B1: $R=\sqrt{109}$ Do not allow decimal equivalents
M1: Allow for $\tan\alpha=\pm\frac{3}{10}$
A1: $\alpha=16.70°$

(b)(i)
B1: see scheme
(b)(ii)
B1ft: their $11+$ their $\sqrt{109}$ Allow decimals here.

(c)
M1: Sets $80t+"16.70"=540$. Follow through on their 16.70
M1: Solves their $80t+"16.70"=540$ correctly to find t
A1: $t=6$ mins 32 seconds

(d)
B1: States that to increase the speed of the wheel the 80's in the equation would need to be increased.

Q2.

Question	Scheme	Marks	AOs
(a)	$D = 5 + 2\sin(30 \times 6.5)° = \text{awrt } 4.48\text{m}$ with units	B1	3.4
		(1)	
(b)	$3.8 = 5 + 2\sin(30t)° \Rightarrow \sin(30t)° = -0.6$	M1 A1	1.1b 1.1b
	$t = 10.77$	dM1	3.1a
	10:46 a.m. or 10:47 a.m.	A1	3.2a
		(4)	
			(5 marks)

Notes:

(a)

B1: Scored for using the model ie. substituting $t = 6.5$ into $D = 5 + 2\sin(30t)°$ and stating $D = awrt\ 4.48\text{m}$. The units must be seen somewhere in (a). So allow when $D = 4.482.. = 4.5\text{ m}$

Allow the mark for a correct answer without any working.

(b)

M1: For using $D = 3.8$ and proceeding to $\sin(30t)° = k$, $|k| \le 1$

A1: $\sin(30t)° = -0.6$ This may be implied by any correct answer for t such as $t = 7.2$

If the A1 implied, the calculation must be performed in degrees.

dM1: For finding **the first value** of t for their $\sin(30t)° = k$ after $t = 8.5$.

You may well see other values as well which is not an issue for this dM mark

(Note that $\sin(30t)° = -0.6 \Rightarrow 30t = 216.9°$ as well but this gives $t = 7.2$)

For the correct $\sin(30t)° = -0.6 \Rightarrow 30t = 323.1 \Rightarrow t = \text{awrt } 10.8$

For the incorrect $\sin(30t)° = +0.6 \Rightarrow 30t = 396.9 \Rightarrow t = \text{awrt } 13.2$

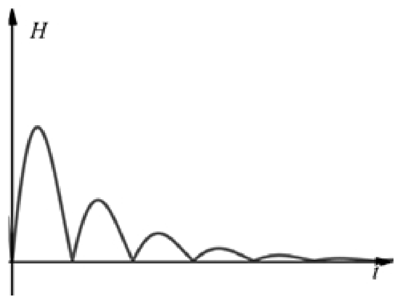
So award this mark if you see $30t = \text{invsin } their -0.6$ to give the first value of t where $30t > 255$

A1: Allow 10:46 a.m. (12 hour clock notation) or 10:46 (24 hour clock notation) oe

Allow 10:47 a.m. (12 hour clock notation) or 10:47 (24 hour clock notation) oe

DO NOT allow 646 minutes or 10 hours 46 minutes.

Q3.

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$f'(x) = -2.5e^{-0.25x}\sin x + 10e^{-0.25x}\cos x$	M1	This mark is given for a method to differentiate to find an expression for $f'(x)$
		A1	This mark is given for correctly differentiating to find an expression for $f'(x)$
	$f'(x) = 0$ $\Rightarrow e^{-0.25x}(-2.5\sin x + 10\cos x) = 0$ $\Rightarrow (-2.5\sin x + 10\cos x) = 0$	M1	This mark is given for setting $f'(x) = 0$ and finding as method to solve for $\tan x$
	$\frac{\sin x}{\cos x} = \frac{10}{2.5}$ $\tan x = 4$	A1	This mark is given for showing that $\tan x = 4$ as required.
(b)		M1	This mark is given for a graph with a correct shape
		A1	This mark is given for a graph with heights > 0
(c)	$\tan x = 4$, $x = 1.326$ $t = \pi + 1.326 = 4.47$	M1	This method is given for finding a value for t between the first and second bounce
	$H(4.47) = \lvert 10e^{-0.25\times 4.47}\sin 4.47\rvert$	M1	This mark is given for substituting the value of $t = \pi + \arctan 4$ into $H(t)$
	$= \lvert 3.27 \times -0.97\rvert$ $= 3.17$ metres	A1	This mark is given for finding the maximum height of the ball
(d)	The time between each bounce should not stay the same when the heights of each bounce are getting smaller	B1	This mark is given for a valid explanation of why the model should not be used the predict the time of each bounce

Q4.

Question	Scheme	Marks	AOs
(a)	$R=\sqrt{5}$	B1	1.1b
	$\tan\alpha = 2 \Rightarrow \alpha = ...$	M1	1.1b
	$\alpha = 1.107$	A1	1.1b
		(3)	
	$\theta = 5+\sqrt{5}\sin\left(\frac{\pi t}{12}+1.107-3\right)$		
(b)	$\left(5+\sqrt{5}\right)$ °C or awrt 7.24°C	B1ft	2.2a
		(1)	
(c)	$\frac{\pi t}{12}+1.107-3=\frac{\pi}{2} \Rightarrow t=$	M1	3.1b
	$t=$ awrt 13.2	A1	1.1b
	Either 13:14 or 1:14 pm or 13 hours 14 minutes after midnight.	A1	3.2a
		(3)	
			(7 marks)
Notes:			

(a)

B1: $R=\sqrt{5}$ only.

M1: Proceeds to a value of α from $\tan\alpha=\pm2, \tan\alpha=\pm\frac{1}{2}, \sin\alpha=\pm\frac{2}{"R"}$ OR $\cos\alpha=\pm\frac{1}{"R"}$

It is implied by either awrt 1.11 (radians) or 63.4 (degrees)

A1: $\alpha=$ awrt 1.107

(b)

B1ft: Deduces that the maximum temperature is $(5+\sqrt{5})$ °C or awrt 7.24 °C Remember to isw

Condone a lack of units. Follow through on their value of R so allow $(5+"R")$ °C

(c)

M1: An complete strategy to find t from $\frac{\pi t}{12}\pm1.107-3=\frac{\pi}{2}$.

Follow through on their 1.107 but the angle must be in radians.

It is possible via degrees but only using $15t\pm63.4-171.9=90$

A1: awrt $t=13.2$

A1: The question asks for the time of day so accept either 13:14, 1:14 pm, 13 hours 14 minutes after midnight, 13h 14, or 1 hour 14 minutes after midday. If in doubt use review

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It is possible to attempt parts (b) and (c) via differentiation but it is unlikely to yield correct results.

$$\frac{d\theta}{dt}=\frac{\pi}{12}\cos\left(\frac{\pi t}{12}-3\right)-\frac{2\pi}{12}\sin\left(\frac{\pi t}{12}-3\right)=0 \Rightarrow \tan\left(\frac{\pi t}{12}-3\right)=\frac{1}{2} \Rightarrow t=13.23=13:14$$ scores M1 A1 A1

$$\frac{d\theta}{dt}=\cos\left(\frac{\pi t}{12}-3\right)-2\sin\left(\frac{\pi t}{12}-3\right)=0 \Rightarrow \tan\left(\frac{\pi t}{12}-3\right)=\frac{1}{2} \Rightarrow t=13.23=13:14$$ they can score M1 A0 A1 (SC)

A value of $t=1.23$ implies the minimum value has been found and therefore incorrect method M0.

Q5.

Question	Scheme	Marks	AOs
(a)	$R=\sqrt{5}$	B1	1.1b
	$\tan\alpha=\frac{1}{2}$ or $\sin\alpha=\frac{1}{\sqrt{5}}$ or $\cos\alpha=\frac{2}{\sqrt{5}}\Rightarrow\alpha=...$	M1	1.1b
	$\alpha=0.464$	A1	1.1b
		(3)	
(b)(i)	$3+2\sqrt{5}$	B1ft	3.4
(ii)	$\cos(0.5t+0.464)=1\Rightarrow 0.5t+0.464=2\pi$ $\Rightarrow t=...$	M1	3.4
	$t=11.6$	A1	1.1b
		(3)	
(c)	$3+2\sqrt{5}\cos(0.5t+0.464)=0$ $\cos(0.5t+0.464)=-\frac{3}{2\sqrt{5}}$	M1	3.4
	$\cos(0.5t+0.464)=-\frac{3}{2\sqrt{5}}\Rightarrow 0.5t+0.464=\cos^{-1}\left(-\frac{3}{2\sqrt{5}}\right)$ $\Rightarrow t=2\left(\cos^{-1}\left(-\frac{3}{2\sqrt{5}}\right)-0.464\right)$	dM1	1.1b
	So the time required is e.g.: $2(3.977...-0.464)-2(2.306...-0.464)$	dM1	3.1b
	$=3.34$	A1	1.1b
		(4)	
(d)	e.g. the "3" would need to vary	B1	3.5c
		(1)	
		(11 marks)	

Notes
(a)
B1: $R=\sqrt{5}$ only.
M1: Proceeds to a value for α from $\tan\alpha=\pm\frac{1}{2}$ or $\sin\alpha=\pm\frac{1}{"R"}$ or $\cos\alpha=\pm\frac{2}{"R"}$
It is implied by either awrt 0.464 (radians) or awrt 26.6 (degrees)
A1: $\alpha=$awrt 0.464
(b)(i)
B1ft: For $(3+2\sqrt{5})$ m or awrt 7.47 m and remember to isw. Condone lack of units.
Follow through on their R value so allow $3+2\times$Their R. (Allow in decimals with at least 3sf accuracy)
(b)(ii)
M1: Uses $0.5t\pm"0.464"=2\pi$ to obtain a value for t
Follow through on their 0.464 but this angle must be in radians.
It is possible in degrees but only using $0.5t\pm"26.6"=360$
A1: Awrt 11.6

Alternative for (b):

$$H = 3 + 4\cos(0.5t) - 2\sin(0.5t) \Rightarrow \frac{dH}{dt} = -2\sin(0.5t) - \cos(0.5t) = 0$$

$$\Rightarrow \tan(0.5t) = -\frac{1}{2} \Rightarrow 0.5t = 2.677..., 5.819... \Rightarrow t = 5.36,\ 11.6$$

$$t = 11.6 \Rightarrow H = 7.47$$

Score as follows:

M1: For a complete method:

Attempts $\frac{dH}{dt}$ and attempts to solve $\frac{dH}{dt} = 0$ for t

A1: For $t =$ awrt 11.6

B1ft: For awrt 7.47 or 3 + 2× Their R

(c)

M1: Uses the model and sets $3 + 2"\sqrt{5}"\cos(...) = 0$ and proceeds to $\cos(...) = k$ where $|k| < 1$.

Allow e.g. $3 + 2"\sqrt{5}"\cos(...) < 0$

dM1: Solves $\cos(0.5t \pm "0.464") = k$ where $|k| < 1$ to obtain at least one value for t

This requires e.g. $2\left(\pi + \cos^{-1}(k) \pm \tan^{-1}\left(\frac{1}{2}\right)\right)$ or e.g. $2\left(\pi - \cos^{-1}(k) \pm \tan^{-1}\left(\frac{1}{2}\right)\right)$

Depends on the previous method mark.

dM1: A fully correct strategy to find the required duration. E.g. finds 2 consecutive values of t when $H = 0$ and subtracts. Alternatively finds t when H is minimum and uses the times found correctly to find the required duration.

Depends on the previous method mark.

Examples:

Second time at water level – first time at water level:

$$2\left(\pi + \cos^{-1}\left(\frac{3}{2\sqrt{5}}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right) - 2\left(\pi - \cos^{-1}\left(\frac{3}{2\sqrt{5}}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right) = 7.02685... - 3.68492...$$

2×(first time at minimum point – first time at water level):

$$2\left(2\left(\pi - \tan^{-1}\left(\frac{1}{2}\right)\right) - 2\left(\pi - \cos^{-1}\left(\frac{3}{2\sqrt{5}}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right)\right) = 2(5.35589... - 3.68492...)$$

Note that both of these examples equate to $4\cos^{-1}\left(\frac{3}{2\sqrt{5}}\right)$ which is not immediately obvious but may be seen as an overall method.

There may be other methods – if you are not sure if they deserve credit send to review.

A1: Correct value. Must be 3.34 (not awrt).

Special Cases in (c):

Note that if candidates have an incorrect α and have e.g. $3 + 2\sqrt{5}\cos(0.5t - 0.464)$, this has no impact on the final answer. So for candidates using $3 + 2\sqrt{5}\cos(0.5t \pm \alpha)$ in (c) allow all the marks including the A mark as a correct method should always lead to 3.34

Some values to look for:

$$0.5t \pm "0.464" = \pm 2.306,\ \pm 3.977,\ \pm 8.598,\ \pm 10.26$$

(d)

B1: Correct refinement e.g. As in scheme. If they suggest a specific function to replace the "3" then it must be sensible e.g. a trigonometric function rather than e.g. a quadratic/linear one.