1. (a) Prove that

$$1 - \cos 2\theta \equiv \tan\theta \sin 2\theta, \quad \theta \neq \frac{(2n+1)\pi}{2}, \quad n \in \mathbb{Z}$$

**(3)**

(b) Hence solve, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, the equation

$$(\sec^2 x - 5)(1 - \cos 2x) = 3\tan^2 x \sin 2x$$

Give any non-exact answer to 3 decimal places where appropriate.

**(6)**

a) $RHS = \tan\theta \sin 2\theta \equiv \frac{\sin\theta}{\cancel{\cos\theta}}(2\sin\theta\cancel{\cos\theta})$

$\equiv 2\sin^2\theta$

but $\cos 2\theta \equiv 1 - 2\sin^2\theta \quad \therefore 2\sin^2\theta \equiv 1 - \cos 2\theta$ //

hence $RHS \equiv 2\sin^2\theta \equiv 1 - \cos 2\theta$
$= LHS$

b) $(\sec^2 x - 5)(1 - \cos 2x) = 3\tan^2 x \sin 2x$

$(\sec^2 x - 5)(\cancel{\tan x \sin 2x}) = 3\tan x(\cancel{\tan x \sin 2x})$

$\sec^2 x - 5 = 3\tan x$ $\quad \sec^2 x \equiv 1 + \tan^2 x$

$1 + \tan^2 x - 5 = 3\tan x$

$\tan^2 x - 3\tan x - 4 = 0$

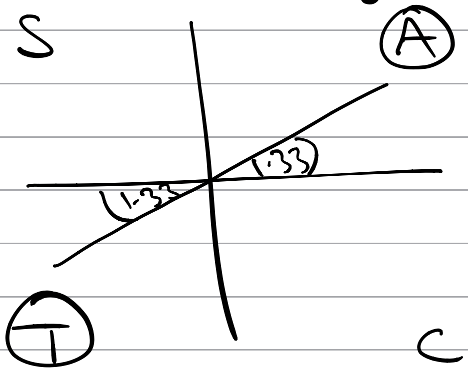
$(\tan x - 4)(\tan x + 1) = 0$

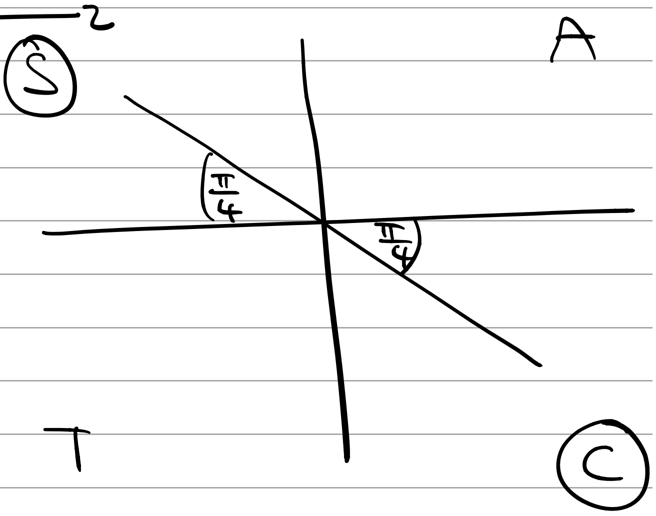
$\tan x = 4 \quad , \quad \tan x = -1$

$x = \tan^{-1}(4) = 1.326 \quad , \quad x = \tan^{-1}(-1) = \frac{-\pi}{4}$

**Question 1 continued**

range: $-\frac{\pi}{2} < x < \frac{\pi}{2}$





$\boxed{x = 1.326}$

$\boxed{x = -\frac{\pi}{4}}$

**2.** (a) Prove

$$\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} \equiv 2\cot 2\theta \qquad \theta \neq (90n)°,\ n \in \mathbb{Z}$$

**(4)**

(b) Hence solve, for $90° < \theta < 180°$, the equation

$$\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} = 4$$

giving any solutions to one decimal place.

**(3)**

a) $\dfrac{\cos 3\theta}{\sin\theta} + \dfrac{\sin 3\theta}{\cos\theta} = 2\cot 2\theta$

Trig Identities :

- $\cos(a-b) = \cos a\cos b + \sin a \sin b$
  $\hookrightarrow a = 3\theta,\ b = \theta$

$\Rightarrow \dfrac{\cos 3\theta\cos\theta + \sin 3\theta\sin\theta}{\sin\theta\cos\theta}$ ①

$\Rightarrow \dfrac{\cos(3\theta-\theta)}{\sin\theta\cos\theta} = \dfrac{\cos(2\theta)}{\sin\theta\cos\theta}$ ①

- $\sin 2\theta = 2\sin\theta\cos\theta$
  $\Rightarrow \frac{1}{2}\sin 2\theta = \sin\theta\cos\theta$
- $\dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta} = \cot\theta$

① $= \dfrac{\cos(2\theta)}{\frac{1}{2}\sin(2\theta)} = \dfrac{2}{\tan 2\theta}$

$= 2\cot 2\theta$

$\Rightarrow \dfrac{\cos 3\theta}{\sin\theta} + \dfrac{\sin 3\theta}{\cos\theta} \equiv \underline{\underline{2\cot 2\theta}}$ as required. ①

b) $\dfrac{\cos 3\theta}{\sin\theta} + \dfrac{\sin 3\theta}{\cos\theta} = 4$

Part a : $\dfrac{\cos 3\theta}{\sin\theta} + \dfrac{\sin 3\theta}{\cos\theta} = 2\cot 2\theta = \dfrac{2}{\tan 2\theta}$

$\Rightarrow 2\cot 2\theta = 4 \Rightarrow \dfrac{2}{\tan 2\theta} = 4$ ① $\Rightarrow \tan 2\theta = \frac{1}{2}$

$\Rightarrow 2\theta = \tan^{-1}(\frac{1}{2})$ ①

$\swarrow 180 + 26.6$

$\Rightarrow 2\theta = 26.6°$ and $2\theta = 206.6°$

$\Rightarrow \theta = 13.3°$ and $\theta = \underline{\underline{103.3°}}$

$\Rightarrow 13.3°$ is not a valid solution since out of range $\Rightarrow \theta = \underline{\underline{103.3°}}$ ①

**3.** **In this question you should show all stages of your working.**

**Solutions relying entirely on calculator technology are not acceptable.**

(a) Given that $1 + \cos 2\theta + \sin 2\theta \neq 0$ prove that

$$\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} \equiv \tan\theta$$

**(4)**

(b) Hence solve, for $0 < x < 180°$

$$\frac{1 - \cos 4x + \sin 4x}{1 + \cos 4x + \sin 4x} = 3\sin 2x$$

giving your answers to one decimal place where appropriate.

**(4)**

(a) $$\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta} \equiv \tan\theta$$

①  ①

$$\frac{1 - (1 - 2\sin^2\theta) + 2\sin\theta\cos\theta}{1 + (2\cos^2\theta - 1) + 2\sin\theta\cos\theta} \equiv \tan\theta$$

$$\frac{1 - 1 + 2\sin^2\theta + 2\sin\theta\cos\theta}{2\cos^2\theta + 2\sin\theta\cos\theta} = \tan\theta$$

① $$\frac{\cancel{2}\sin\theta\,\cancel{(\sin\theta + \cos\theta)}}{\cancel{2}\cos\theta\,\cancel{(\cos\theta + \sin\theta)}} \equiv \tan\theta$$

① $$\frac{\sin\theta}{\cos\theta} \equiv \tan\theta$$

$$\tan\theta \equiv \tan\theta$$ ⋕

**Question continued**

(b) Let $\theta = 2x$

$\tan 2x = 3\sin 2x$ ①

$\frac{\sin 2x}{\cos 2x} = 3\sin 2x$

$\sin 2x = 3\sin 2x\cos 2x$

$\sin 2x - 3\sin 2x\cos 2x = 0$

$0^\circ < x < 180^\circ$

$0^\circ < 2x < 360^\circ$

$\sin 2x(1 - 3\cos 2x) = 0$

$\sin 2x = 0$ $\quad 1 - 3\cos 2x = 0$ ①

$\cos 2x = \frac{1}{3}$

$\sin 2x = 0$ : $2x = 180^\circ$ ↗ we don't include 0 because of the range $0 < 2x < 360$

$\therefore x = 90^\circ$

$\cos 2x = 1/3$ : $2x = 70.52\ldots, 289.47\ldots$

$x = 35.26\ldots, 144.736\ldots$

$x \approx 35.3^\circ, 144.7^\circ$

$\therefore x = 35.3^\circ, 90^\circ, 144.7^\circ$ (1dp) ①

① ✱

**4.** (a) Prove that

$$\tan\theta + \cot\theta \equiv 2\operatorname{cosec}2\theta, \qquad \theta \neq \frac{n\pi}{2}, n \in \mathbb{Z}$$

**(4)**

(b) Hence explain why the equation

$$\tan\theta + \cot\theta = 1$$

does not have any real solutions.

**(1)**

a) $\tan\theta + \cot\theta \equiv 2\operatorname{cosec}2\theta$

$\equiv \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$ ① $\equiv \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$ ①

$\equiv \frac{1}{\frac{1}{2}\sin 2\theta}$ ① $\equiv \frac{2}{\sin 2\theta}$

$\equiv 2\operatorname{cosec}2\theta$

$\Rightarrow \tan\theta + \cot\theta \equiv 2\operatorname{cosec}2\theta$ as required. ①

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

$\sin^2\theta + \cos^2\theta = 1$

$\frac{1}{2}\sin(2\theta) = \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}$

$\operatorname{cosec}2\theta \equiv \frac{1}{\sin 2\theta}$

(b) Hence explain why the equation

$$\tan\theta + \cot\theta = 1$$

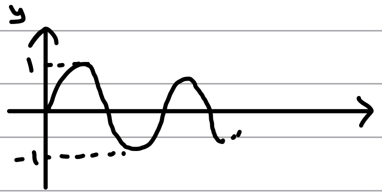
does not have any real solutions.

(1)

$\tan\theta + \cot\theta \equiv 2\operatorname{cosec}(2\theta) \equiv \frac{2}{\sin 2\theta} = 1$

$\Rightarrow \sin 2\theta = 2$

$2\theta = \sin^{-1}(2)$



$-1 \le \sin 2\theta \le 1$ ①

$2 > 1$

$\Rightarrow$ No Solution exists.

**(Total for Question is 5 marks)**

**5.** (i) Solve, for $0 \leqslant x < \frac{\pi}{2}$, the equation

$$4\sin x = \sec x$$

**(4)**

(ii) Solve, for $0 \leqslant \theta < 360°$, the equation

$$5\sin\theta - 5\cos\theta = 2$$

giving your answers to one decimal place.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

**(5)**

i) $\sec(x) = \frac{1}{\cos(x)}$

$4\sin(x) = \frac{1}{\cos(x)}$ ✓ 1

$4\sin(x)\cos(x) = 1$

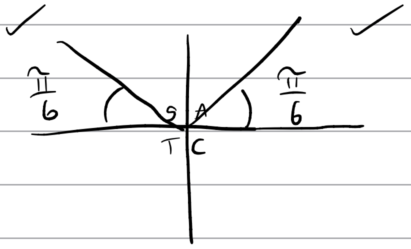
$2\sin(x)\cos(x) = \sin(2x)$

$2\sin(x)\cos(x) = \frac{1}{2}$ ✓ 1

$0 \le x < \frac{\pi}{2}$

$0 \le 2x < \pi$

$\sin(2x) = \frac{1}{2}$



$2x = \sin^{-1}\left(\frac{1}{2}\right)$
$= \frac{\pi}{6}, \frac{5\pi}{6}$

$x = \frac{\pi}{12}, \frac{5\pi}{12}$ ✓✓ 2

(ii) Solve, for $0 \leqslant \theta < 360°$, the equation

$$5\sin\theta - 5\cos\theta = 2$$

giving your answers to one decimal place.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

**(5)**

ii) $\sin(\theta - \alpha) = \sin(\theta)\cos(\alpha) - \sin(\alpha)\cos(\theta)$

$R\sin(\theta - \alpha) = R\sin(\theta)\cos(\alpha) - R\sin(\alpha)\cos(\theta)$

$R\sin(\theta)\cos(\alpha) - R\sin(\alpha)\cos(\theta) = 5\sin(\theta) - 5\cos(\theta)$ ✓ 1

$\underline{\sin(\theta)}$: $R\cos(\alpha) = 5$ [1]

$\underline{\cos(\theta)}$: $\cancel{-}R\sin(\alpha) = \cancel{-}5$ [2]

$[2] \div [1] \rightarrow$ $\dfrac{\cancel{R}\sin(\alpha)}{\cancel{R}\cos(\alpha)} = \dfrac{5}{5}$

$\tan(\alpha) = 1$

$\alpha = 45°$ ✓ 1

$[2]^2 + [1]^2 \rightarrow$ $R^2\sin^2(\alpha) + R^2\cos^2(\alpha) = 5^2 + 5^2$

$R^2[\underbrace{\sin^2(\alpha) + \cos^2(\alpha)}_{} ] = 50$ $\overset{1}{}$

$\sin^2(\alpha) + \cos^2(\alpha) = 1$

$R^2 = 50$

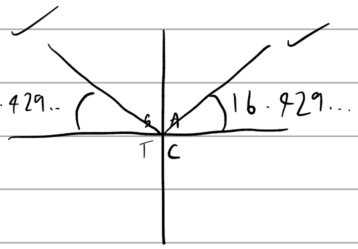
$R = \sqrt{50} = 5\sqrt{2}$ ✓ 1

$5\sqrt{2}\sin(\theta - 45) = 2$

$\sin(\theta - 45) = \frac{\sqrt{2}}{5}$

$\theta - 45 = 16.4299..., 163.57...$ ✓ 1

$0 \leq \theta \leq 360$



$\underline{\theta = 61.4, 206.6}$ (1 d.p) ✓ 1

**6.** (a) Express $\sin x + 2\cos x$ in the form $R\sin(x+\alpha)$ where $R$ and $\alpha$ are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of $R$ and give the value of $\alpha$ in radians to 3 decimal places.

**(3)**

a) $\sin x + 2\cos x \longrightarrow R\sin(x+\alpha)$

$\boxed{1}$ Find $\alpha$

$\boxed{2}$ Find $R$

$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha \Rightarrow \sin x = R\sin x\cos\alpha \Rightarrow R\cos\alpha = 1$

$2\cos x = R\cos x\sin\alpha \Rightarrow R\sin\alpha = 2$

$\Rightarrow \tan\alpha = \frac{2}{1} \Rightarrow \alpha = \tan^{-1}(2)$

$\alpha = 1.10714\ldots$ ① $\Rightarrow \alpha = \underline{1.107}$ (3 d.p) ①

$R = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$ ①

$\Rightarrow \alpha = \underline{1.107}$ (radians), $R = \underline{\sqrt{5}}$ $\Rightarrow \sin x + 2\cos x = \underline{\sqrt{5}}\sin(x + 1.107)$

The temperature, $\theta$°C, inside a room on a given day is modelled by the equation

$$\theta = 5 + \sin\left(\frac{\pi t}{12} - 3\right) + 2\cos\left(\frac{\pi t}{12} - 3\right) \qquad 0 \leqslant t < 24$$

where $t$ is the number of hours after midnight.

Using the equation of the model and your answer to part (a),

(b) deduce the maximum temperature of the room during this day,

**(1)**

b) $\theta = 5 + \sin\left(\frac{\pi t}{12} - 3\right) + 2\cos\left(\frac{\pi t}{12} - 3\right)$

let $x = \frac{\pi t}{12} - 3$, we can use our answer from part a. $\left(\sqrt{5}\sin(x + 1.107) = \sin x + 2\cos x\right)$

$\Rightarrow \theta = 5 + \sqrt{5}\sin\left(\frac{\pi t}{12} - 3 + 1.107\right)$, we have a maximum when $\sin x = 1$

$\Rightarrow \theta = (5 + \sqrt{5})^\circ C$ or $\theta = \underline{7.24}$ °C (3 s.f) ①

(c) find the time of day when the maximum temperature occurs, giving your answer to the nearest minute.

**(3)**

c) $\theta = 5 + \sqrt{5}\sin\left(\frac{\pi t}{12} - 3 + 1.107\right)$

In part b, we said the maximum temperature occurs when $\sin x = 1$.

$\Rightarrow x = \sin^{-1}(1)$

$\Rightarrow \underline{x = \pi/2}$

$\Rightarrow \frac{\pi t}{12} - 3 + 1.107 = \frac{\pi}{2}$

$\Rightarrow \frac{\pi t}{12} = \frac{\pi}{2} + 3 - 1.107$

0.2 of an hour is equal $0.2 \times 60 = 12$ mins

$\Rightarrow \cancel{\pi}t = \frac{12\left(\frac{\pi}{2} + 3 - 1.107\right)}{\pi}$ ① $\Rightarrow t = 13.2$ hours ①

$\Rightarrow t = 13$ hours and 12 minutes after midnight. ①

**7.** (a) Show that

$$\operatorname{cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta \qquad \theta \neq (180n)° \qquad n \in \mathbb{Z}$$

**(3)**

a) $\text{Cosec}\,\theta - \text{Sin}\,\theta \equiv \frac{1}{\text{Sin}\,\theta} - \text{Sin}\,\theta$

$\equiv \frac{1 - \text{Sin}^2\theta}{\text{Sin}\,\theta}$ ①

$\equiv \frac{\text{Cos}^2\theta}{\text{Sin}\,\theta}$

$\equiv \text{Cos}\,\theta \cdot \frac{\text{Cos}\,\theta}{\text{Sin}\,\theta}$

$\text{Cosec}\,\theta - \text{Sin}\,\theta \equiv \underline{\text{Cos}\,\theta\,\text{Cot}\,\theta}$ as required. ①

$\text{Cosec}\,\theta = \frac{1}{\text{Sin}\,\theta}$ ①

$\text{Sin}^2\theta + \text{Cos}^2\theta = 1$

$\Rightarrow 1 - \text{Sin}^2\theta = \text{Cos}^2\theta$

$\frac{\text{Cos}\,\theta}{\text{Sin}\,\theta} = \text{Cot}\,\theta$

(b) Hence, or otherwise, solve for $0 < x < 180°$

$$\operatorname{cosec}x - \sin x = \cos x\cot(3x - 50°)$$

**(5)**

b) Part a : $\text{Cosec}\,\theta - \text{Sin}\,\theta \equiv \text{Cos}\,\theta\,\text{Cot}\,\theta$

$\Rightarrow \underline{\text{Cos}\,x}\,\text{Cot}\,x = \underline{\text{Cos}\,x} \cdot \text{Cot}\,(3x - 50°)$

$\div \text{Cos}\,x$ $\qquad \div \text{Cos}\,x$

$\Rightarrow \underbrace{\underline{\text{Cot}}\,x = \underline{\text{Cot}}\,(3x - 50°)}_{\text{'equal'}} \Rightarrow x = 3x - 50$ ①

$2x = 50°$

$\underline{x = 25°}$ ①

Since cotx has a period of 180°, we can find a second solution

$\Rightarrow x + 180 = 3x - 50$ ①

$\Rightarrow 2x = 230° \Rightarrow \underline{x = 115°}$ ①

There will be a third solution when $\text{Cos}\,x = 0 \Rightarrow x = \text{Cos}^{-1}(0)$

$\Rightarrow \underline{x = 90°}$ ①

$\Rightarrow \underline{x = 25°}, \ \underline{x = 90°}$ and $\underline{x = 115°}$

8. (a) Show that

$$\cos 3A \equiv 4\cos^3 A - 3\cos A$$

**(4)**

$\cos \overset{2A \quad A}{3A} = \cos(2A + A)$

**Compound Angle Formula:**

- $\cos(x+y) = \cos x \cos y - \sin x \sin y$

We can use the compound angle formula with $x = 2A$ and $y = A$.

**Double Angle Formula:**

- $\cos 2A = 2\cos^2 A - 1$
- $\sin 2A = 2\sin A \cos A$
- $\sin^2 x + \cos^2 x = 1.$

$\Rightarrow 2\sin^2 x = 2 - 2\cos^2 x$ ←

$\Rightarrow \cos 3A = \cos 2A \cos A - \sin 2A \sin A$ ①

$= (2\cos^2 A - 1)\cos A - (2\sin A\cos A)\sin A$ ①

$= 2\cos^3 A - \cos A - 2\sin^2 A\cos A$

$= 2\cos^3 A - \cos A - (2 - 2\cos^2 A)\cos A$ ①

$= \underline{2\cos^3 A} - \cos A - 2\cos A + \underline{2\cos^3 A}$

$\Rightarrow \cos 3A \equiv 4\cos^3 A - 3\cos A$ ① as required.

(b) Hence solve, for $-90° \leqslant x \leqslant 180°$, the equation

$$1 - \cos 3x = \sin^2 x$$

**(4)**

$1 - \cos 3x = \sin^2 x$

- $\sin^2 x + \cos^2 x = 1$
- Part a : $\cos 3A \equiv 4\cos^3 A - 3\cos A$

$\Rightarrow 1 - \cos 3x = 1 - \cos^2 x$

$\Rightarrow 1 - (4\cos^3 x - 3\cos x) = 1 - \cos^2 x$

let $\cos x = y$

$\Rightarrow -4y + y + 3$

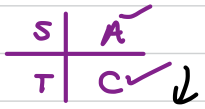
$-(4y - y - 3)$

$-(4y+3)(y-1)$

$\Rightarrow 1 - 4\cos^3 x + 3\cos x - 1 + \cos^2 x = 0 \Rightarrow \cos^2 x + 3\cos x - 4\cos^3 x = 0$ ①

$\Rightarrow \cos x(\underline{\cos x + 3 - 4\cos^2 x}) = 0$

$\Rightarrow \underline{\cos x}(4\cos x + 3)(\cos x - 1) = 0$



$\Rightarrow \cos x = 0$

$x = 90°$

$x = -90$ ①

$4\cos x + 3 = 0$

$\cos x = -\frac{3}{4} \Rightarrow x = 139°$

$\cos x - 1 = 0$ ①

$x = 0$

$\Rightarrow$ Solutions are : $x = -90°, 0°, \underline{90°}$ and $139°$. ①