Trigonometry and Modelling Cheat Sheet

Pure Year 2

This chapter builds upon the previous, introducing more useful methods, formulae and identities relating to trigonometric functions

Addition Formulae

- $\sin(A+B) \equiv sinAcosB + cosAsinB$
- $sin(A-B) \equiv sinAcosB - cosAsinB$
- $\cos(A+B) \equiv cosAcosB - sinAsinB$
- $cos(A-B) \equiv cosAcosB + sinAsinB$
- $\tan(A+B) \equiv \dfrac{tanA + tanB}{1 - tanAtanB}$
- $tan(A-B) \equiv \dfrac{tanA - tanB}{1 + tanAtanB}$

You need to know how to use the above formulae to find exact values of trigonometric functions for various angles.

Example 1: Show, using the formula for $\sin(A+B)$, that $\sin(75°) = \frac{\sqrt{6}+\sqrt{2}}{4}$

We can rewrite $\sin(75°)$ as $\sin(45° + 30°)$. We choose 45 and 30 because we know the exact values of $\sin(45°)$, $\cos(45°)$, $\sin(30°)$ and $\cos(30°)$, so when we put them into the addition formula, we will have all terms given as exact values.

$\sin(75°) \equiv \sin(45+30)$

$\sin(45+30) \equiv sin45cos30 + cos45sin30$

$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

Double-angle formulae

- $\sin(2A) \equiv 2sinAcosA$
- $\cos(2A) \equiv cos^2A - sin^2A = 1 - 2sin^2A = 2cos^2A - 1$
- $\tan(2A) \equiv \dfrac{2tanA}{1 - tan^2A}$

You can be asked to reproduce these proofs!

Example 2: Using the addition formulae, prove each of the above double-angle formulae.

Proving the double-angle sine formula:	$\sin(2A) = \sin(A+A) = sinAcosA + cosAsinA = 2sinAcosA$
Proving the double-angle cosine formula:	$\cos(2A) = \cos(A+A) = cosAcosA - sinAsinA = cos^2A - sin^2A$
Using $sin^2A + cos^2A \equiv 1$ to prove the other cosine double angle formulae:	By replacing cos^2A with $1 - sin^2A$: $\Rightarrow \cos(2A) = 1 - 2sin^2A$. Also, by replacing sin^2A with $1 - cos^2A$: $\Rightarrow \cos(2A) = 2cos^2A - 1$
Proving the double-angle tangent formula:	$\tan(2A) = \tan(A+A) = \dfrac{tanA + tanA}{1 - tanAtanA} = \dfrac{2tanA}{1 - tan^2A}$

You can see that there are three different versions for the cosine double angle formula. It is important you are familiar with all three as one may be more useful than the others in certain questions.

Example 3: Simplify as much as possible the expression: $sin^4x - 2sin^2xcos^2x + cos^4x$

Spotting the factorisation:	$sin^4x - 2sin^2xcos^2x + cos^4x = (cos^2x - sin^2x)^2$
Using $cos2x = cos^2x - sin^2x$:	$= (cos2x)^2 = cos^22x$

Example 4: Simplify as much as possible the expression: $\sqrt{1 + cosx}$

Since $cos2x = 2cos^2x - 1$	$cosx = 2cos^2\left(\frac{x}{2}\right) - 1$
Substituting this result into the given expression:	$\Rightarrow \sqrt{1+cosx} = \sqrt{1 + 2cos^2\left(\frac{x}{2}\right) - 1} = \sqrt{2cos^2\left(\frac{x}{2}\right)}$

Simplifying $asinx \pm bcosx$

Expressions of the above form can be simplified into one trigonometric term.

- $asinx \pm bcosx$ can be expressed as $Rsin(x \pm \alpha)$
- $acosx \pm bsinx$ can be expressed as $Rcos(x \mp \alpha)$

When the coefficient of sin is positive, use $Rsin(x \pm \alpha)$ and when the coefficient of cos is positive, use $Rcos(x \mp \alpha)$. Of course, when both coefficients are positive then you can use either form.

where $a, b, R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

The procedure for achieving the above simplifications can be broken down into three steps:

[1] Expand the form using the addition formulae, and equate it to $asinx \pm bcosx$

[2] Compare the coefficients of $sinx$ and $cosx$ on both sides of the equation, to get two equations in terms of R and α.

[3] Solve these simultaneously to find R and α.

Example 5: Express $cos2x - 2sin2x$ in the form $Rcos(2x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Proving the double-angle sine formula:	$1cos2x - 2sin2x = Rcos(2x+\alpha) \equiv Rcos2x\,cos\alpha - Rsin2x\,sin\alpha$
Equating coefficients:	$1 = Rcos\alpha$ (1) (equating $cos2x$ coefficients); $-2 = -Rsin\alpha$ (2) (equating $sin2x$ coefficients)
Solving simultaneously.: We divide equation [2] by [1].	$tan\alpha = \dfrac{Rsin\alpha}{Rcos\alpha} = \dfrac{-2}{1} = -2$; $\therefore \alpha = \arctan(-2) = 1.11$
Finding R: Square equations [1] and [2] then add them together. We also use the identity $cos^2\alpha + sin^2\alpha \equiv 1$	$(1)^2 + (2)^2 \Rightarrow R^2cos^2\alpha + R^2sin^2\alpha = (1)^2 + (-2)^2$; $\Rightarrow R^2(cos^2\alpha + sin^2\alpha) = 5$; $\Rightarrow R^2 = 5 \therefore R = \sqrt{5}$
Putting everything together:	So $cos2x - 2sin2x = \sqrt{5}\cos(2x + 1.11)$

A shortcut for finding R is to use $R = \sqrt{a^2 + b^2}$

This form is often useful because it makes solving equations and finding minimum/maximum values much easier.

Example 6: Given that $g(x) = \dfrac{18}{50 + cos2x - 2sin2x}$,

calculate: (i) the maximum value of $g(x)$.
(ii) The smallest positive value of x at which this minimum occurs.

Proving the double-angle sine formula:	$g(x) = \dfrac{18}{50 + \sqrt{5}\cos(2x + 1.11)}$
The maximum value of $g(x)$ occurs when the denominator is at a minimum. We can deduce that the denominator is a minimum when $\cos(2x + 1.11)$ is at a minimum. i.e. when $\cos(2x+1.11) = -1$, since $-1 \le \cos(2x+1.11) \le 1$.	$\therefore g(x)_{max} = \dfrac{18}{50 + \sqrt{5}(-1)} = \dfrac{18}{50 - \sqrt{5}}$
From (i), we established that the maximum value of $g(x)$ occurs when $\cos(2x+1.11) = -1$. Therefore, we need to solve for the smallest positive value of x such that this is true. Since $\cos(x)$ has its first positive minimum at $x = \pi$, our minimum will be found by solving the equation $2x + 1.11 = \pi$. Alternatively, we can use CAST or a graphical method to solve $\cos(2x+1.11) = -1$.	$\pi = 2x_{min} + 1.11$; $\therefore x_{min} = \dfrac{\pi - 1.11}{2} = 1.02$

Solving equations

To solve more complicated trigonometric expressions, you will first need to simplify the equation using the formulae and methods we have covered so far. Here is an example showing how we do this in practice:

Example 7: Solve $3\sin(x - 45°) - \sin(x + 45°) = 0$ in the interval $0 \le x \le 360°$

Using the addition formulae	$3sinxcos45 - 3cosxsin45 - sinxcos45 - cosxsin45 = 0$
Simplifying	$2sinxcos45 - 4cosxsin45 = 0$
Using exact values for $cos45°, sin45°$	$\sqrt{2}sinx - 2\sqrt{2}cosx = 0$
Dividing through by $\sqrt{2}$	$sinx = 2cosx$
Dividing through by $cosx$ and finding the principal solution:	$tanx = 2 \therefore x = \arctan(2) = 63.4°$
Using CAST or a graphical method, we can find all the solutions:	The solutions in the given interval are: $x = 63.4°, 243.4°$.

Proving identities

You need to be able to use everything we have covered so far to prove identities. You must start from one side of the equation and use your knowledge of trigonometric identities to manipulate the expression and achieve what is on the other side.

There is no set procedure to follow in your manipulation. Your knowledge of the identities is being tested, so you need to make sure you are very familiar with the content in this chapter and the previous. As with most of Mathematics, the most useful preparation tool here is practice.

Example 8: Show that $cos^4x = \frac{3}{8} + \frac{1}{2}cos2x + \frac{1}{8}cos4x$

Starting with the LHS:	$LHS = (cos^2x)(cos^2x)$
Using the double-angle cosine identity to express cos^2x in terms of $cos2x$:	Since $cos2x = 2cos^2x - 1 \Rightarrow cos^2x = \left(\dfrac{cos2x + 1}{2}\right)$
Substituting this result back into the LHS:	$\Rightarrow LHS = \left(\dfrac{cos2x+1}{2}\right)\left(\dfrac{cos2x+1}{2}\right)$
Expanding:	$= \frac{1}{4}(cos^22x + 2cos2x + 1)$
Using the double-angle cosine identity again to express cos^22x in terms of $cos4x$.	Since $cos2x = 2cos^2x - 1 \Rightarrow cos^22x = \left(\frac{cos4x+1}{2}\right)$
Substituting this result back into the LHS:	$\frac{1}{4}\left[\dfrac{cos4x+1}{2} + 2cos2x + 1\right]$
Simplifying to achieve the RHS:	$= \frac{1}{8}cos4x + \frac{1}{2}cos2x + \frac{1}{8} + \frac{1}{4}$; $= \frac{1}{8}cos4x + \frac{1}{2}cos2x + \frac{3}{8} = RHS$

Modelling with trigonometric functions

In the exam you will likely be given problems where trigonometric functions are used to model real-life situations, often involving the forms $Rsin(x \pm \alpha)$ and $Rcos(x \pm \alpha)$. To succeed in these questions, you must properly understand the scenario given to you. Read through the text more than once to make sure you understand what is going on. The maths itself is the same as before; you just need to be able to apply it in the context of the question.

Example 9: A town wishes to build a large Ferris wheel to be used as a tourist attraction. The height above the ground, H metres, of a passenger on the Ferris wheel is modelled by the equation

$$H = 25 + 20\sin\left(\frac{2}{5}t\right) - 65\cos\left(\frac{2}{5}t\right),$$

where H is the height of the passenger above the ground and t is the number of minutes after the ride has started. The angles are given in radians.

a) By rewriting H in the form $A + Rcos\left(\frac{2}{5}t + \alpha\right)$ where A, R, α are positive constants, find the maximum height of the Ferris wheel above the ground.

b) Find the time taken for one complete revolution.

Use the method from example 4 to simplify the two trigonometric terms into one term. Note that since we need to use the $Rcos\left(\frac{2}{5}t + \alpha\right)$ form, our cosine coefficient must be positive. So consider $65\cos\left(\frac{2}{5}t\right) - 20\sin\left(\frac{2}{5}t\right)$ rather than $20\sin\left(\frac{2}{5}t\right) - 65\cos\left(\frac{2}{5}t\right)$	$65\cos\left(\frac{2}{5}t\right) - 20\sin\left(\frac{2}{5}t\right) \equiv 5\sqrt{185}\cos\left(\frac{2}{5}t + 0.298\right)$; $\therefore H = 25 - 5\sqrt{185}\cos\left(\frac{2}{5}t + 0.298\right)$
By looking at our equation, we can deduce that H is maximum when $\cos\left(\frac{2}{5}t + 0.298\right)$ is minimum.	H_{max} occurs when $\cos\left(\frac{2}{5}t + 0.298\right) = -1$. $\therefore H_{max} = 25 + 5\sqrt{185}$ = max height above ground
This question is essentially asking us to calculate the period of our function H. To do so, we just need to look at our cosine function: since $cos(t)$ has a period of 2π, we can conclude that H has a period of $\frac{2\pi}{\frac{2}{5}} = 5\pi$. The reason we can say this is because the cosine term in H is $\cos\left(\frac{2}{5}t + 0.298\right)$. This tells us that compared to $\cos(t)$, all the t values are multiplied by $\frac{1}{\frac{2}{5}}$, so our time period is also multiplied by $\frac{1}{\frac{2}{5}}$ giving us 5π.	The time taken for one complete revolution is $\frac{2\pi}{\frac{2}{5}} = 5\pi$.