Radians Cheat Sheet

Radians are simply another way to measure angles. In many areas of mathematics and physics, using radians as opposed to degrees turns out to be much more convenient. For example, the arc length formula which we will cover soon is greatly simplified by using radians.

Using Radians

To convert between radians and degrees, you can use the fact that:

- $1\, radian = \frac{180°}{\pi}$

It helps to remember the following angles in radians:

- $30° = \frac{\pi}{6}$
- $45° = \frac{\pi}{4}$
- $60° = \frac{\pi}{3}$
- $90° = \frac{\pi}{2}$
- $180° = \pi$
- $360° = 2\pi$

You can be asked to sketch trigonometric functions giving your angles in radians, so you should be very efficient at converting between radians and degrees.

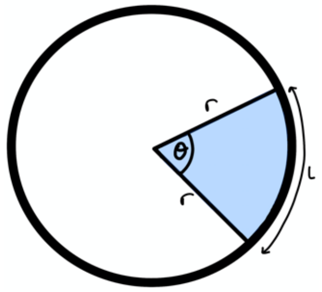
You also need to learn the exact value of certain trigonometric ratios given in radians:

- $sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$
- $sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$
- $sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
- $cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$
- $cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
- $cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
- $tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$
- $tan\left(\frac{\pi}{3}\right) = \sqrt{3}$
- $tan\left(\frac{\pi}{4}\right) = 1$

Arc length

To find the arc length l of a sector of a circle, we can use the formula

- $l = r\theta$
  where r is the radius of the circle and θ is the angle contained in the sector, **given in radians**.



Areas

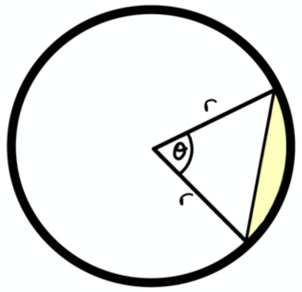
To find the area of a sector, shown in blue, we can use the formula

- $A = \frac{1}{2}r^2\theta$

To find the area of a segment, shown in yellow, we can use the formula

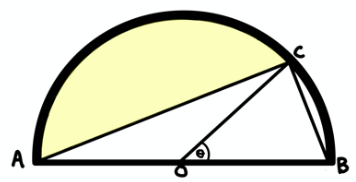
- $A = \frac{1}{2}r^2(\theta - sin\theta)$

recall that the area of a triangle is $\frac{1}{2}absin\theta$



You need to be able to apply the above formulae to problems.

Example 1: In the diagram below, AB is the diameter of a circle centre O of radius r cm and $\angle BOC = \theta$ radians. Given that the area of $\triangle\, COB$ is equal to that of the shaded segment, show that $\theta + 2sin\theta = \pi$.



Area $COB = \frac{1}{2}r^2 sin\theta$ (area of a triangle, $OC = OB = r$)

Shaded segment $= \frac{1}{2}r^2((\pi - \theta) - sin\,(\pi - \theta))$

Equating: $\frac{1}{2}r^2 sin\theta = \frac{1}{2}r^2((\pi - \theta) - sin(\pi - \theta))$

Notice that $sin(\pi - \theta) = sin(\theta)$,
dividing through by $\frac{1}{2}r^2$ and rearranging: $sin\theta = \pi - \theta - sin\theta$
$\therefore \theta + 2sin\theta = \pi$ as required

Solving trigonometric equations

You also need to be able to solve trigonometric equations using radians. The method is exactly the same as with degrees, but you need to give your answers in radians.

- If the interval is given in radians, then you should leave your answers in radians.

Let's go through an example.

Example 2: Solve $8tan2x = 7$ in the interval $0 \le x \le 2\pi$.

The interval is given in radians, so we must make sure we work in radians. Don't forget to switch to the radians mode on your calculator.

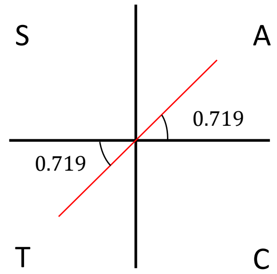
$tan2x = \frac{7}{8}$

$2x = \arctan\left(\frac{7}{8}\right) = 0.719$

Using CAST or a graphical method, our solutions are:

$2x = 0.719, \pi + 0.719$
$\Rightarrow 2x = 0.719, 3.86$

$\therefore x = 0.36, 1.93$ are our final solutions.



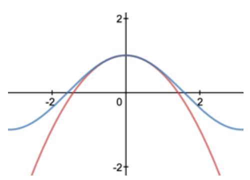
As you can see, the procedure is still the same. You just need to be comfortable with using radians.

Small angle approximations

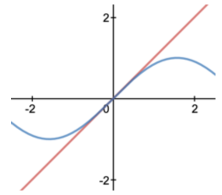
When θ is close to zero and measured in radians, we can use the following approximations:

- $sin\theta \approx \theta$
- $cos\theta \approx 1 - \frac{\theta^2}{2}$
- $tan\theta \approx \theta$

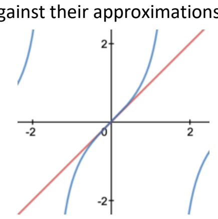
To see why these are good approximations, we can graph each of the functions against their approximations.



$y = cosx$
$y = 1 - \frac{x^2}{2}$



$y = sinx$
$y = x$



$y = tanx$
$y = x$

We can see that around $x = 0$, the graphs are almost identical. This explains why these approximations are suitable for x close to zero. There is no set range for which these approximations are to be used.

Example 3: When θ is close to zero, show that $\frac{4cos3\theta - 2 + 5sin\theta}{1 - sin2\theta}$ can be rewritten as $9\theta + 2$.

Inputting our approximations:

$\Rightarrow 4cos3\theta - 2 + 5sin\theta \approx 4\left(1 - \frac{(3\theta)^2}{2}\right) - 2 + 5\theta = 2 + 5\theta - 18\theta^2$

$\Rightarrow 1 - sin2\theta \approx 1 - 2\theta$

$\therefore \frac{4cos3\theta - 2 + 5sin\theta}{1 - sin2\theta} \approx \frac{2 + 5\theta - 18\theta^2}{1 - 2\theta} = \frac{(1 - 2\theta)(2 + 9\theta)}{1 - 2\theta} = 2 + 9\theta$ as required.

Once you substitute your approximations, it becomes a matter of simplifying your expression to achieve the desired result.

Example 4: When θ is close to zero, find the approximate value of $cos^4\theta - sin^4\theta$.

$\Rightarrow cos^4\theta - sin^4\theta = (cos^2\theta + sin^2\theta)(cos^2\theta - sin^2\theta)$ — difference of two squares
$= (1)(cos^2\theta - sin^2\theta)$ — using $cos^2\theta + sin^2\theta = 1$
$\Rightarrow [1 - sin^2\theta - sin^2\theta] = 1 - 2sin^2\theta = 1 - 2\theta^2$

Exam-style question

We will now go through an exam-style question where you can be expected to use what you have learnt so far.

Example 5: The diagram shows the cross section $ABCD$ of a glass prism. $AD = BC = 4\ cm$ and both are at right angles to DC. AB is the arc of a circle, centre O and radius 6 cm. Given that $\angle AOB = 2\theta$ radians, and that the perimeter of the cross section is $2(7 + \pi)$ cm,

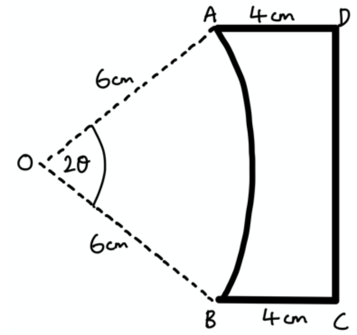
a) Show that $(2\theta + 2sin\theta - 1) = \frac{\pi}{3}$.

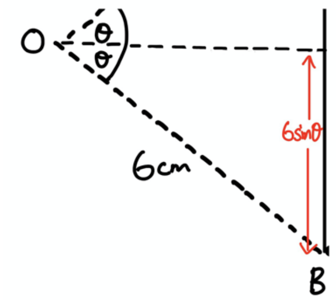
b) Verify that $\theta = \frac{\pi}{6}$,

c) Find the area of the cross-section.

d) Show that when x is small,

the expression $\frac{\pi(2(1 - cosx) - sin^2x + sinx - 1)}{12tanx - 12} \approx \frac{1}{2}\theta$.





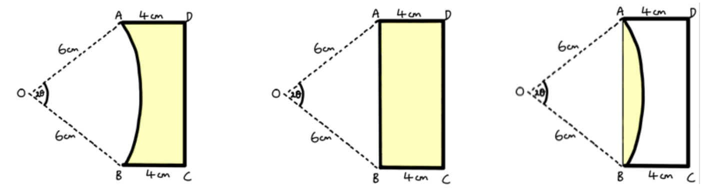
a) Perimeter $= 4 + 4 + DC + AB$
$AB = r\theta = 6 \times 2\theta = 12\theta$

Using trigonometry, we can see that $DC = 2(6sin\theta)$
$\therefore$ Perimeter $= 12sin\theta + 8 + 12\theta = 2(7 + \pi)$

Rearranging: $12\theta + 12sin\theta - 6 = 2\pi$
Dividing by 6 gives: $2\theta + 2sin\theta - 1 = \frac{\pi}{3}$ as required.

b) We just need to plug in $\theta = \frac{\pi}{6}$ to the LHS and check it is equal to the RHS.
$\Rightarrow 2\left(\frac{\pi}{6}\right) + 2sin\frac{\pi}{6} - 1 = \frac{\pi}{3} + 1 - 1 = \frac{\pi}{3} = RHS$ so $\theta = \frac{\pi}{6}$.

c) *Area of cross section = Area of rectangle ABCD – Area of segment AB*



Area of rectangle ABCD $= 4 \times 12sin\left(\frac{\pi}{6}\right) = 24$
Area of segment AB $= \frac{1}{2}(r)^2(2\theta - sin2\theta) = \frac{1}{2}(6)^2\left(\frac{\pi}{3} - sin\frac{\pi}{3}\right) = 18\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = 6\pi - 9\sqrt{3}$

$\Rightarrow Area\ of\ cross\ section = 24 - (6\pi - 9\sqrt{3}) = 24 + 9\sqrt{3} - 6\pi$

e) Starting with the numerator:

$\pi(2(1 - cosx) - sin^2x + sinx - 1) \approx \pi\left[2\left(1 - \left(1 - \frac{x^2}{2}\right)\right) - (x)^2 + x - 1\right]$

$\Rightarrow \pi\left[2\left(\left(\frac{x^2}{2}\right)\right) - (x)^2 + x - 1\right] = \pi[x^2 - x^2 + x - 1] = \pi(x - 1)$

Now considering the denominator:

$12tanx - 12 \approx 12x - 12 = 12(x - 1)$

$\therefore \frac{\pi(2(1 - cosx) - sin^2x + sinx - 1)}{12tanx - 12} = \frac{\pi(x - 1)}{12(x - 1)} = \frac{\pi}{12}$

But in part b, we verified $\theta = \frac{\pi}{6}$.

$\frac{\pi}{12} = \frac{1}{2} \times \frac{\pi}{6}$ hence. $\frac{\pi(2(1 - cosx) - sin^2x + sinx - 1)}{12tanx - 12} \approx \frac{1}{2}\theta$ as required.