Trigonometric functions Cheat Sheet

Reciprocal trigonometric functions

Previously, you have met three trigonometric functions; $sinx, cosx$ and $tanx$.

This chapter introduces three more trigonometric functions, known as the reciprocal trigonometric functions:

- $secx = \frac{1}{cosx}$ (undefined for values of x for which $cosx = 0$)
- $cosecx = \frac{1}{sinx}$ (undefined for values of x for which $sinx = 0$)
- $cotx = \frac{1}{tanx}$ (undefined for values of x for which $tanx = 0$)

Since division by zero is undefined, we have that these functions are undefined when the denominators are equal to zero.

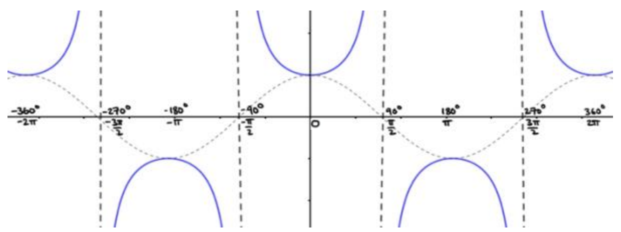
Note that $cotx = \frac{1}{tanx} = \frac{cosx}{sinx}$, simply by replacing $tanx$ with $\frac{sinx}{cosx}$. This will sometimes be a more useful form to use.

Careful: It is not true that: $secx = (cosx)^{-1}$, $cosecx = (sinx)^{-1}$, $cotx = (tanx)^{-1}$
The negative power has a different meaning when used with trigonometric functions.

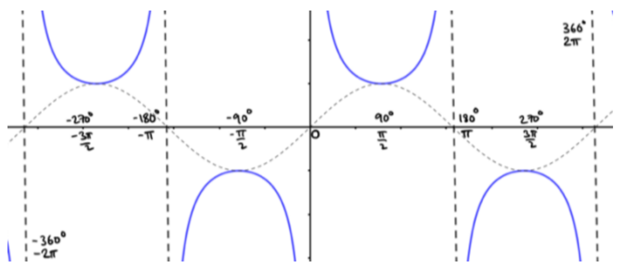
Graphing the reciprocal functions

You need to be able to sketch the reciprocal trigonometric functions as well as any transformations, using radians and degrees. Below are the graphs of the reciprocal functions

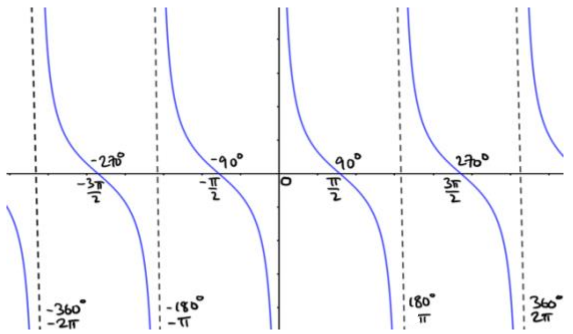
$y = sec x$ Dotted curve represents $y = cosx$
Domain: $x \in \mathbb{R}, x \neq odd\ multiple\ of\ \frac{\pi}{2}$ (90°)
Range: $y \leq -1, y \geq 1$



$y = cosecx$ Dotted curve represents $y = sinx$
Domain: $x \in \mathbb{R}, x \neq multiple\ of\ \pi$ (180°)
Range: $y \leq -1$ or $y \geq 1$



$y = cotx$
Domain: $x \in \mathbb{R}, x \neq multiple\ of\ \pi$ (180°)
Range: $y \in \mathbb{R}$



Reciprocal trigonometric identities

Recall from Pure Year 1, that $sin^2x + cos^2x = 1$ [1]

Taking [1], let us divide through by sin^2x:

$$\frac{sin^2x}{sin^2x} + \frac{cos^2x}{sin^2x} = \frac{1}{sin^2x}$$

$$1 + cot^2x = cosec^2x$$

We can also divide [1] through by cos^2x:

$$\frac{sin^2x}{cos^2x} + \frac{cos^2x}{cos^2x} = \frac{1}{cos^2x}$$

$$tan^2x + 1 = sec^2x$$

This gives us the following identities:

- $1 + cot^2x = cosec^2x$
- $1 + tan^2x = sec^2x$

You could be asked to prove these identities, so make sure you are able to reproduce the proofs on the left.

Simplifying expressions and proving identities

You can use the definitions and identities we have covered so far to simplify and prove expressions involving the reciprocal trig functions.

There is no trick or standard procedure to be used for these questions. Your ability to manipulate trigonometric expressions using reciprocal functions and identities is being tested, so the most useful thing you can do is properly familiarise yourself with these functions and the above identities. As with most of mathematics, the most useful tool here is practice.

When proving identities, you must start from one side and work your way towards the other side. You can start from any side, so pick whichever seems like an easier starting point.

Example 1: Prove that $sec^2x + cosec^2x = sec^2x\,cosec^2x$

Starting from the LHS, we have:	$LHS = sec^2x + cosec^2x$
using the tan^2x and sec^2x identities:	$(1 + tan^2x) + (1 + cot^2x) = 2 + tan^2x + cot^2x$
Rewriting $tanx$ as $\frac{sinx}{cosx}$ and $cotx$ as $\frac{cosx}{sinx}$:	$= 2 + \frac{sin^2x}{cos^2x} + \frac{cos^2x}{sin^2x}$
combining everything into one fraction:	$= \frac{sin^4x + cos^4x}{sin^2xcos^2x} + 2$ $= \frac{sin^4x + cos^4x + 2sin^2xcos^2x}{sin^2xcos^2x}$
using $sin^2x + cos^2x = 1$	$= \frac{(sin^2x + cos^2x)^2}{sin^2xcos^2x} = \frac{1}{sin^2xcos^2x}$
splitting the fraction up into a product, giving us the RHS	$= \frac{1}{sin^2x} \cdot \frac{1}{cos^2x} = sec^2x\,cosec^2x = RHS$

Solving equations

Previously, in Pure Year 1, you learnt how to solve trigonometric equations involving $sinx, cosx$ and $tanx$. Now we will look at solving equations that also involve the reciprocal functions. The only difference here is that you need to use the identities and definitions we have covered in this chapter in order to simplify the equation, before you can solve it.

Example 2: Solve the equation $secx = \sqrt{2}$ in the interval $0 \leq x \leq 360°$

rewriting $secx$ as $\frac{1}{cosx}$:	$\frac{1}{cosx} = \sqrt{2}$
taking the reciprocal of both sides:	$\therefore cosx = \frac{1}{\sqrt{2}}$
We can solve this via CAST or a graphical method giving:	$x = 45°, 315°$.

Example 3: Solve $sec^2x = 3tanx$ in the interval $0 \leq x \leq 360°$.

using $1 + tan^2x = sec^2x$	$1 + tan^2x = 3tanx$
rearranging	$tan^2x - 3tanx + 1 = 0$
Solving the quadratic:	By the quadratic formula, $tanx = \frac{3+\sqrt{5}}{2}$, $tanx = \frac{3-\sqrt{5}}{2}$
We have two equations to solve. Using CAST or a graphical method, our solutions are:	$x = 20.9°, 69.1°, 201°, 249°$ to 3 s.f.

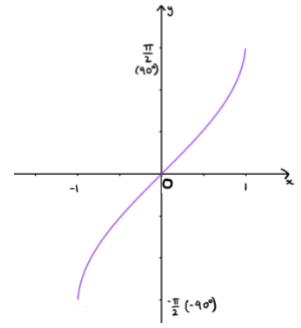
Inverse trigonometric functions

A function only has an inverse if it is one-to-one. The trigonometric functions aren't one-to-one by definition, but if we restrict the domains, we can turn them into one-to-one functions. This allows us to define the inverse functions, which we can sketch by reflecting the $sinx, cosx$ and $tanx$ graphs in the line $y = x$.

Reflecting $y = sinx$ in the line $y = x$ using the domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ gives us its inverse function, $arcsinx$:

$y = arcsinx$

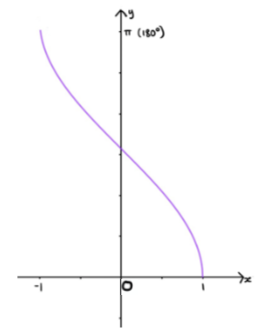
Domain: $-1 \leq x \leq 1$
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ or $-90° \leq y \leq 90°$



Reflecting $y = cosx$ in the line $y = x$ using the domain $0 \leq x \leq \pi$ gives us its inverse function, $arccosx$:

$y = arccosx$

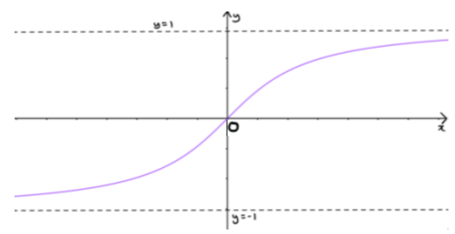
Domain: $-1 \leq x \leq 1$
Range: $0 \leq y \leq \pi$ or $0 \leq y \leq 180°$



Reflecting $y = tanx$ in the line $y = x$ using the domain: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ gives us its inverse function, $arctanx$:

$y = arctanx$

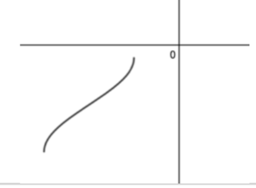
Domain: $x \in \mathbb{R}$
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ or $-90° \leq y \leq 90°$



Remember that since these functions are inverses, we have that $arcsinx(sinx) = sin(arcsinx) = x$. Of course, this works for $arccosx$ and $arctan$x too, not just $arcsinx$.

Just like with the reciprocal functions, you may be asked to sketch a transformation of any of the inverse functions, or even to solve an equation involving an inverse function.

Example 4: a) Sketch the graph of $y = g(x)$ where $g(x) = arcsin(x + 2) - 2$
b) Find the value of x, to 2 decimal places, for which $3g(x + 1) + \pi = 0$

Starting from the LHS, we have:	
using the tan^2x and sec^2x identities:	$3g(x + 1) + \pi = 3[arcsin(x + 1 + 2) - 2] + \pi$
Rewriting $tanx$ as $\frac{sinx}{cosx}$ and $cotx$ as $\frac{cosx}{sinx}$:	$\Rightarrow 3arcsin(x + 3) + \pi - 6 = 0$ $arcsin(x + 3) = \frac{6-\pi}{3}$
combining everything into one fraction:	$\Rightarrow x + 3 = sin(\frac{6-\pi}{3})$
using $sin^2x + cos^2x = 1$	$\Rightarrow x = sin(\frac{6-\pi}{3}) - 3 = -2.18$