SEQUENCES AND SERIES — *Answers*

1 **a** $a = 108,\ ar^3 = 32$

$\therefore\ r^3 = 32 \div 108 = \frac{8}{27}$

$r = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$

$u_3 = 108 \times (\frac{2}{3})^2 = 48$

b $S_\infty = \dfrac{108}{1-\frac{2}{3}} = 324$

2 $= 1 + 5(-2x) + 10(-2x)^2 + 10(-2x)^3 + 5(-2x)^4 + (-2x)^5$

$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

3 **a** new subscribers in 4th week

$= 200 \times (1.15)^3 = 304.175$

$= 304$ (nearest unit)

b new subscribers: GP, $a = 200$, $r = 1.15$

$S_{10} = \dfrac{200[(1.15)^{10} - 1]}{1.15 - 1} = 4060.74$

total no. of subscribers $= 3600 + S_{10}$

$= 7661$ (nearest unit)

4 **a** $= 1 + 7(4x) + \frac{7\times 6}{2}(4x)^2 + \ldots$

$= 1 + 28x + 336x^2 + \ldots$

b $(1 + 2x)^2(1 + 4x)^7$

$= (1 + 4x + 4x^2)(1 + 28x + 336x^2 + \ldots)$

term in x^2

$= (1)(336x^2) + (4x)(28x) + (4x^2)(1)$

coefficient of $x^2 = 336 + 112 + 4 = 452$

5 **a** $= 1 + 2n(\frac{x}{k}) + \frac{2n(2n-1)}{2}(\frac{x}{k})^2 + \frac{2n(2n-1)(2n-2)}{3\times 2}(\frac{x}{k})^3 + \ldots$

$= 1 + \dfrac{2n}{k}x + \dfrac{n(2n-1)}{k^2}x^2 + \dfrac{2n(n-1)(2n-1)}{3k^3}x^3 + \ldots$

b $\dfrac{2n(n-1)(2n-1)}{3k^3} = \dfrac{1}{2} \times \dfrac{n(2n-1)}{k^2}$

$4n(n-1)(2n-1) = 3kn(2n-1)$

$n(2n-1)[4(n-1) - 3k] = 0$

$n > 1\ \therefore\ 4(n-1) - 3k = 0$

$3k = 4(n-1)$

c $\dfrac{2n}{k} = 2\ \therefore\ n = k$

$\therefore\ 3k = 4k - 4$

$k = 4,\ n = 4$

6 **a** $r = 3\sqrt{2} \div \sqrt{6} = \sqrt{3}$

$a = \sqrt{6} \div \sqrt{3} = \sqrt{2}$

b $S_8 = \dfrac{\sqrt{2}[(\sqrt{3})^8 - 1]}{\sqrt{3} - 1}$

$= \dfrac{80\sqrt{2}}{\sqrt{3}-1} \times \dfrac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \dfrac{80\sqrt{2}(\sqrt{3}+1)}{3-1}$

$= 40\sqrt{2}(\sqrt{3}+1)$

7 $\sum_{r=1}^{9} 3^r$: GP, $a = 3$, $r = 3$

$S_9 = \dfrac{3(3^9 - 1)}{3 - 1} = 29523$

$\therefore\ \sum_{r=1}^{9}(3^r - 1) = 29523 - 9$

$= 29\,514$

8 **a** $= 1 + 9(2x) + \frac{9\times 8}{2}(2x)^2 + \frac{9\times 8\times 7}{3\times 2}(2x)^3 + \ldots$

$= 1 + 18x + 144x^2 + 672x^3 + \ldots$

b $(1 - 2x)^9 = 1 - 18x + 144x^2 - 672x^3 + \ldots$

$\therefore\ (1 + 2x)^9 + (1 - 2x)^9$

$= (1 + 18x + 144x^2 + 672x^3 + \ldots) + (1 - 18x + 144x^2 - 672x^3 + \ldots)$

$= 2 + 288x^2$ (ignoring terms in x^4 and higher)

c let $x = 0.001$

$\therefore\ 1.002^9 + 0.998^9 \approx 2 + 0.000\,288$

$= 2.000\,288$ (7sf)

9 $(k-x)^9 = k^9 + 9(k^8)(-x) + \frac{9\times 8}{2}(k^7)(-x)^2 + \ldots$

$= k^9 - 9k^8x + 36k^7x^2 + \ldots$

$\therefore\ -b = -9k^8$ and $b = 36k^7$

$9k^8 = 36k^7$

$9k^7(k-4) = 0$

$k \neq 0\ \therefore\ k = 4$

$a = k^9 = 262\,144$

$b = 9k^8 = 589\,824$

10 $= 3^4 + 4(3)^3(2x) + 6(3)^2(2x)^2 + 4(3)(2x)^3 + (2x)^4$

$= 81 + 216x + 216x^2 + 96x^3 + 16x^4$

11 **a** $\frac{t}{1-r} = 3t$

$1 - r = \frac{t}{3t} = \frac{1}{3}\ \therefore\ r = \frac{2}{3}$

b $\frac{t[1-(\frac{2}{3})^4]}{1-\frac{2}{3}} = 130$

$t = (\frac{1}{3} \times 80) \div \frac{65}{81} = 54$

12 **a** $= 1 + 4(-2x) + 6(-2x)^2 + 4(-2x)^3 + (-2x)^4$

$= 1 - 8x + 24x^2 - 32x^3 + 16x^4$

b let $x = y^2 - 2y$

$(1 + 4y - 2y^2)^4$

$= 1 - 8(y^2 - 2y) + 24(y^2 - 2y)^2 + \ldots$

term in $y^2 = -8y^2 + 24(-2y)^2$

coefficient of $y^2 = -8 + 96 = 88$

13 **a** $= 12000 \times (0.75)^4$

$= 3796.875$

$= £3800$ (3sf)

b GP: $a = 12000$, $r = 0.75$

$S_8 = \frac{12000[1-(0.75)^8]}{1-0.75}$

$= £43\,200$ (3sf)

14 **a** $p(-2) = 1^4 - (-1)^4 = 1 - 1 = 0$

$\therefore\ (x+2)$ is a factor of $p(x)$

b $p(x) = [x^4 + 4(x^3)(3) + 6(x^2)(3^2) + 4(x)(3^3) + 3^4] - [x^4 + 4x^3 + 6x^2 + 4x + 1]$

$= 8x^3 + 48x^2 + 104x + 80$

$= 8(x^3 + 6x^2 + 13x + 10)$

$$
\begin{array}{r|l}
 & x^2 + 4x + 5 \\
x + 2 & x^3 + 6x^2 + 13x + 10 \\
 & x^3 + 2x^2 \\
 & \quad 4x^2 + 13x \\
 & \quad 4x^2 + 8x \\
 & \qquad 5x + 10 \\
 & \qquad 5x + 10
\end{array}
$$

$p(x) = 8(x+2)(x^2 + 4x + 5)$

c $8(x+2)(x^2 + 4x + 5) = 0$

$x = -2$ or $(x^2 + 4x + 5) = 0$

$b^2 - 4ac = 16 - 20 = -4$

$b^2 - 4ac < 0\ \therefore$ no real sols to $(x^2 + 4x + 5) = 0$

$\therefore$ only one real solution to $p(x) = 0$

15 **a** $(1 - x)(1 + 2x)^n$

$= (1 - x)[1 + n(2x) + \frac{n(n-1)}{2}(2x)^2 + \ldots]$

$= (1 - x)[1 + 2nx + 2n(n - 1)x^2 + \ldots]$

$\therefore\ 2n(n - 1) - 2n = 198$
$n^2 - 2n - 99 = 0$
$(n + 9)(n - 11) = 0$
$n \geq 0\ \ \therefore\ n = 11$

b $(1 - x)(1 + 2x)^{11}$
$= (1 - x)[\ldots + \frac{11\times10}{2}(2x)^2 + \frac{11\times10\times9}{3\times2}(2x)^3 + \ldots]$
$= (1 - x)[\ldots + 220x^2 + 1320x^3 + \ldots]$
$\therefore$ coefficient of $x^3 = 1320 - 220 = 1100$

16 $= (\frac{3}{x})^4 + 4(\frac{3}{x})^3(-x) + 6(\frac{3}{x})^2(-x)^2$
$+ 4(\frac{3}{x})(-x)^3 + (-x)^4$
$= x^4 - 12x^2 + 54 - \frac{108}{x^2} + \frac{81}{x^4}$

17 **a** $S_4 = 3^4 - 1 = 80$
$S_3 = 3^3 - 1 = 26$

$u_4 = S_4 - S_3 = 80 - 26 = 54$

b $S_{n-1} = 3^{n-1} - 1$
$u_n = S_n - S_{n-1}$
$= (3^n - 1) - (3^{n-1} - 1)$
$= 3^n - 3^{n-1}$
$= 3^n(1 - \frac{1}{3}) = \frac{2}{3}(3^n)$ $\quad [k = \frac{2}{3}]$

c $u_{n-1} = \frac{2}{3}(3^{n-1})$
$u_n \div u_{n-1} = \frac{2}{3}(3^n) \div \frac{2}{3}(3^{n-1}) = 3$
$u_n \div u_{n-1}$ is constant $\therefore$ geometric

18 **a** $3(x - 3) = y - 3$
$y = 3x - 6$

b $\left(\frac{x}{3}\right)^3 = \frac{y}{3}$
$x^3 = 9y = 9(3x - 6)$
$x^3 - 27x + 54 = 0$

c trying $x = 1, 2$ etc. $\Rightarrow$ $x = 3$ is a solution
$\therefore$ $(x - 3)$ is a factor

$$
\begin{array}{r|l}
 & x^2 + 3x - 18 \\
\hline
x - 3 & x^3 + 0x^2 - 27x + 54 \\
 & x^3 - 3x^2 \\
\hline
 & 3x^2 - 27x \\
 & 3x^2 - 9x \\
\hline
 & -18x + 54 \\
 & -18x + 54 \\
\hline
\end{array}
$$

$(x - 3)(x^2 + 3x - 18) = 0$
$(x - 3)(x + 6)(x - 3) = 0$
$x = -6$ or 3