Binomial expansion Cheat Sheet

Using the binomial expansion

The binomial expansion can be used to find accurate approximations of expressions raised to high powers.

In Pure Year 1, you learnt how to expand $(a+bx)^n$ where n is a positive integer and a, b being any constants. We will now learn how to expand a greater range of expressions.

To expand $(a+bx)^n$ when n is no longer a positive integer, we need to use another form of the binomial expansion:

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots$$

This expansion is only valid when $|x| < 1$

- If we wish to expand an expression of the form $(1+bx)^n$, then we can use the above formula by replacing every x with bx.

Example 1: Find the expansion of $\sqrt{1-2x}$ up to and including the term in x^2, and state of values for x for which the expansion is valid.

$\Rightarrow \sqrt{1-2x} = (1-2x)^{\frac{1}{2}}$ Rewriting so the power is visible

$\Rightarrow (1-2x)^{\frac{1}{2}} \approx 1 + \frac{\frac{1}{2}(-2x)}{1!} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)(-2x)^2}{2!}$ Using $n = \frac{1}{2}, x \to -2x$

$\Rightarrow (1-2x)^{\frac{1}{2}} \approx 1 - x - \frac{x^2}{2}$ Simplifying

To find the range of values for which the expansion is valid, we take the "x" term and let $|x| < 1$. Our "x" term here is $-2x$.

So, we let: $|-2x| < 1$

And rearrange for $|x|$: $|2x| < 1 \therefore |x| < \frac{1}{2}$

- If we wish to expand $(a+bx)^n$ where n is not a positive integer, we first need to manipulate our expression into the form $(1+bx)^n$ and then use the above form of the binomial expansion.
- The expansion of $(a+bx)^n$, where n is negative or a fraction, is valid for $|x| < \left|\frac{a}{b}\right|$.

Example 2: Find the series expansion of $(5+4x)^{-2}$ in ascending powers of x, up to and including the term in x^3.

$\Rightarrow (5+4x)^{-2} = \left[5\left(1+\frac{4}{5}x\right)\right]^{-2} = [5]^{-2}\left(1+\frac{4}{5}x\right)^{-2} = \frac{1}{25}\left(1+\frac{4}{5}x\right)^{-2}$

manipulating into the form $(1+bx)^n$

$\left(1+\frac{4}{5}x\right)^{-2} \approx 1 + \frac{(-2)\left(\frac{4}{5}x\right)}{1!} + \frac{-2(-2-1)\left(\frac{4}{5}x\right)^2}{2!} + \frac{-2(-3)(-4)\left(\frac{4}{5}x\right)^3}{3!}$

using $n = -2, x \to \frac{4}{5}x$

$\left(1+\frac{4}{5}x\right)^{-2} \approx 1 - \frac{8}{5}x + \frac{48}{25}x^2 - \frac{256}{125}x^3$

$\therefore (5+4x)^{-2} \approx \frac{1}{25}\left[1 - \frac{8}{5}x + \frac{48}{25}x^2 - \frac{256}{125}x^3\right]$

$\approx \left[\frac{1}{25} - \frac{8}{125}x + \frac{48}{625}x^2 - \frac{256}{3125}x^3\right]$

Harder binomial expansions

Some questions will require you to expand more than one expression and manipulate them. The process is still largely the same as before, there is just more working required.

Example 3: Given that $g(x) = \frac{3}{4-2x} - \frac{2}{3+5x}$, find the first three terms in the series expansion of $g(x)$.

We must expand each term separately, then add the results together.

Rearranging first term into form $(1+bx)^n$:

$\frac{3}{4-2x} = 3(4-2x)^{-1} = 3\left[4\left(1-\frac{1}{2}x\right)\right]^{-1} = 3(4)^{-1}\left(1-\frac{1}{2}x\right)^{-1} = \frac{3}{4}\left(1-\frac{1}{2}x\right)^{-1}$

Now we can expand:

$\frac{3}{4}\left(1-\frac{1}{2}x\right)^{-1} \approx \frac{3}{4}\left[1 + \frac{(-1)}{1!}\left(-\frac{1}{2}x\right) + \frac{-1(-1-1)}{2!}\left(-\frac{1}{2}x\right)^2\right] = \frac{3}{4} + \frac{3}{8}x + \frac{3}{16}x^2$

Rearranging second term into form $(1+bx)^n$:

$-\frac{2}{3+5x} = -2(3+5x)^{-1} = -2\left[3\left(1+\frac{5}{3}x\right)\right]^{-1} = -2(3)^{-1}\left[\left(1+\frac{5}{3}x\right)^{-1}\right] = -\frac{2}{3}\left(1+\frac{5}{3}x\right)^{-1}$

Now we can expand:

$-\frac{2}{3}\left(1+\frac{5}{3}x\right)^{-1} = -\frac{2}{3}\left[1 + \frac{(-1)\left(\frac{5}{3}x\right)}{1!} + \frac{(-1)(-1-1)\left(\frac{5}{3}x\right)^2}{2!}\right] = -\frac{2}{3} + \frac{10}{9}x - \frac{50}{27}x^2$

We have both expansions, so adding them together gives us $g(x)$:

$g(x) \approx \left(\frac{3}{4} + \frac{3}{8}x + \frac{3}{16}x^2\right) + \left(-\frac{2}{3} + \frac{10}{9}x - \frac{50}{27}x^2\right) = \frac{1}{12} + \frac{107}{72}x - \frac{719}{432}x^2.$

Analysing binomial expansions

So far, we have only looked at how to find binomial expansions. Later parts of exam questions will often require you to use your expansion. We will go through three examples displaying the typical style of these questions, and how you can solve them.

Example 4: State the values of x for which the expansion of $(5+4x)^{-2}$ is valid.

From example 2, we saw that

$(5+4x)^{-2} = \frac{1}{25}\left(1+\frac{4}{5}x\right)^{-2} \approx \left[\frac{1}{25} - \frac{8}{125}x + \frac{48}{625}x^2 - \frac{256}{3125}x^3\right]$

Recall that the expansion of $(a+bx)^n$, where n is negative or a fraction, is valid for $|x| < \left|\frac{a}{b}\right|$.

So the range of validity is $|x| < \frac{5}{4}$.

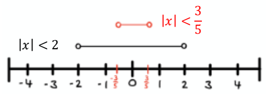
Example 5: Given that $g(x) = \frac{3}{4-2x} - \frac{2}{3+5x}$, find the values of x for which the expansion is valid.

Using the expansion from example 3 and the same method from example 4,

$\frac{3}{4-2x} = 3(4-2x)^{-1} \Rightarrow |x| < \left|\frac{4}{-2}\right| \therefore |x| < 2$

and

$-\frac{2}{3+5x} = -2(3+5x)^{-1} \Rightarrow |x| < \left|\frac{3}{5}\right| \quad so \quad |x| < \frac{3}{5}$



Note that we want where both the inequalities hold! This is when $|x| < \frac{3}{5}$.

Example 6: a) By substituting $x = 0.01$ into the expansion of $\sqrt{1-2x}$, find to 5 decimal places, an approximation for $\sqrt{2}$.
b) Find the percentage error for this approximation.

a) From Example 1, we saw that $\sqrt{1-2x} \approx 1 - x - \frac{x^2}{2}$

We are told to substitute $x = 0.01$, so we start by plugging this into the LHS and RHS:

$\sqrt{1-2(0.01)} \approx 1 - (0.01) - \frac{(0.01)^2}{2}$

$\Rightarrow \sqrt{\frac{49}{50}} \approx 0.98995$ [since $1 - 2(0.01) = \frac{49}{50}$ and putting the RHS into the calculator gives 0.98995]

But we wish to approximate $\sqrt{2}$ not $\sqrt{\frac{49}{50}}$, so we must manipulate the LHS:

$\sqrt{\frac{49}{50}} = \frac{\sqrt{49}}{\sqrt{2x25}} = \frac{7}{5\sqrt{2}} = \frac{7}{10} \times \sqrt{2}$

So $\frac{7}{10} \times \sqrt{2} \approx 0.98995 \quad \therefore \sqrt{2} \approx \frac{0.98995}{\frac{7}{10}} = 1.41421$ to 5 d.p

b) Recall that the formula for percentage error is:

$$\% \text{ error} = \frac{estimate - exact}{exact} \times 100$$

The "estimate" will be our answer to part a and the "exact" will simply be $\sqrt{2}$.

$$\therefore \% \text{ error} = \frac{1.41421 - \sqrt{2}}{\sqrt{2}} \times 100 = -0.00025\% \to 0.00025\%$$

Using partial fractions

You can use partial fractions to simplify more difficult fractions, before using the binomial expansion. Recall that we can split a fraction via partial fractions if there is more than one linear factor in the denominator.

Example 7: Expand $\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)}$ in ascending powers of x as far as the term x^2.

Using the partial fractions method met in Chapter 1, we can find that:

$$\frac{6+7x+5x^2}{(1+x)(1-x)(2+x)} \equiv \frac{2}{1+x} + \frac{3}{1-x} - \frac{4}{2+x}$$

(The working out has been skipped here so revisit Chapter 1 of Pure Year 2 if you are unsure why the above line is true)

We now need to expand each of the above terms separately and add them all together, as we did in example 3.

[1] $\frac{2}{1+x} = 2(1+x)^{-1} = 2\left[1 + \frac{(-1)}{1!}(x) + \frac{(-1)(-1-1)}{2!}(x)^2\right] = 2 - 2x + 2x^2$

[2] $\frac{3}{1-x} = 3(1-x)^{-1} = 3\left[1 + \frac{(-1)}{1!}(-x) + \frac{(-1)(-1-1)}{2!}(-x)^2\right] = 3 + 3x + 3x^2$

[3] $-\frac{4}{2+x} = -4(2+x)^{-1} = -4\left[2\left(1+\frac{x}{2}\right)\right]^{-1} = -4(2)^{-1}\left(1+\frac{x}{2}\right)^{-1} = -2\left(1+\frac{x}{2}\right)^{-1}$

$\Rightarrow -2\left(1+\frac{x}{2}\right)^{-1} \approx -2\left[1 + \frac{(-1)}{1!}\left(\frac{x}{2}\right) + \frac{(-1)(-1-1)}{2!}\left(\frac{x}{2}\right)^2\right] = -2 + x - \frac{x^2}{2}$

Now adding each expansion together:

$$\therefore \frac{6+7x+5x^2}{(1+x)(1-x)(2+x)} \approx [1] + [2] + [3] = (2 - 2x + 2x^2) + (3 + 3x + 3x^2) + \left(-2 + x - \frac{x^2}{2}\right)$$

$$= 3 + 2x + \frac{9}{2}x^2$$