

The binomial expansion can be used to expand brackets raised to large powers. It can be used to simplify probability models with a large number of trials, such as those used by manufacturers to

# Pascal's triangle

predict faults.

You can use Pascal's triangle to quickly expand expressions such as  $(x + 2y)^3$ . Consider the expansions of  $(a + b)^n$  for n = 0,1,2,3 and 4:

$$(a+b)^0 = 1$$

$$(a+b)^1 = 1a + 1b$$

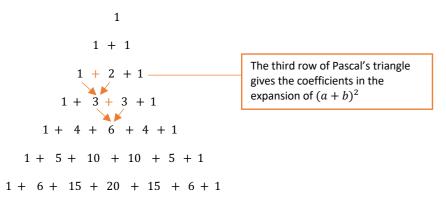
$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$
Each coefficient is the sum of the 2 coefficients immediately above it
$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$
Every term in the expansion of  $(a+b)^n$  has total index n:
$$\ln the 6a^2b^2 term the total index is 2 + 2 = 4.$$

$$\ln the 4ab^3 term the total index is 1 + 3 = 4.$$

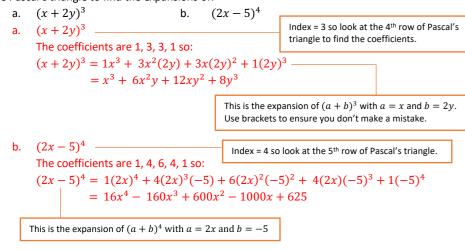
Pascal's triangle is formed by adding adjacent pairs of the numbers to find the numbers on the next row.

Here are the first 7 rows of Pascal's triangle:



The (n+1)th row of Pascal's triangle gives he coefficients in the expansion of  $(a+b)^n$ .

Use Pascal's triangle to find the expansions of:





# Pure Year 1

The coefficient of  $x^2$  in the expansion of of  $(2-cx)^3$  is 294. Find the possible values of the constant c. (Note: if there is an unknown in the expression, form an equation involving the

The coefficients are 1, 3, 3, 1: The term in  $x^2$  is  $3 \times 2(-cx)^2 = 6c^2x^2$ So,  $6c^2 = 294$  $c^2 = 49 \Rightarrow c \pm 7$ 

### **Factorial notation**

Combinations and factorial notation can help you expand binomial expressions. For larger indices, it is quicker than using Pascal's triangle.

Using factorial notation  $3 \times 2 \times 1 = 3!$ 

You can use factorial notation and your calculator to find entries in Pascal's triangle quickly. The number of ways of choosing r items from a group of n items is written as  ${}^{n}C_{r}$  or  ${n \choose r}$ :

$${}^{n}C_{r} = {n \choose r} = \frac{n!}{r!(n-r)!}$$

The rth entry in the nth row of Pascal's triangle is given by  ${}^{n-1}C_{r-1} = {n-1 \choose r-1}$ 

Example 3: Calculate

a. 5!

b.  ${}^{5}C_{2}$ 

c. the 6th entry in the 10th row of Pascal's triangle

a.  $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ 

The *r*th entry in the *n*th row is  $^{n-1}C_{r-1}$ 

b.  ${}^{5}C_{2} = \frac{5!}{2!3!} = \frac{120}{12} = 10$ 

calculator to answer this question. You can calculate  ${}^5C_2$  by using the

Use the  ${}^{n}C_{r}$  and ! functions on your

 $^{n}$   $C_{r}$  function on your calculator.

 ${}^{n}C_{r} = \frac{n!}{r!(n-r)!} = \frac{5!}{2!(5-2)!}$ 

### The binomial expansion

c.  ${}^9C_5 = 126$ 

The binomial expansion is a rule that allows you to expand brackets. You can use  $\binom{n}{r}$  to work out the coefficients in the binomial expansion. For example,

in the expansion of  $(a + b)^5 = (a + b)(a + b)(a + b)(a + b)$ , to find the  $b^3$  term you can choose multiples of b from 3 different brackets. You can do this in  $\binom{5}{2}$  ways so the  $b^3$  term is  $\binom{5}{3}a^2b^3$ .

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n$$

where 
$$\binom{n}{r} = {}^n C_r = \frac{n!}{r!(n-r)!}$$

Example 4: Use the binomial theorem to find the expansion of  $(3-2x)^5$ .

$$(3+2x)^5 = 3^5 + {5 \choose 1}3^4(-2x) + {5 \choose 2}3^3(-2x)^2 + {5 \choose 3}3^2(-2x)^3 + {5 \choose 4}3^1(-2x)^4 + (-2x)^5$$

$$= 243 - 810x + 1080x^2 - 720x^3 + 240x^4 - 32x^5$$

There will be 6 terms. Each term has a total index of 5. Use  $(a + b)^n$ with a = 3, b = -2x and n = 5

## **Solving Binomial Problems**

You can use the general term of the binomial expansion to find individual coefficients in a binomial expansion.

In the expansion of  $(a+b)^n$  the general term is given by  $\binom{n}{r}a^{n-r}b^r$ .

a. Find the coefficient of  $x^4$  in the binomial expansion of  $(2+3x)^{10}$ .  $x^4 \text{ term} = \binom{10}{4} 2^6 (3x)^4$  $= 210 \times 64 \times 81x^4$  $= 1088640x^4$ 

The coefficient of  $x^4$  in the binomial expansion of  $(2 + 3x)^{10}$  is 1088640.

b. Find the coefficient of  $x^3$  in the binomial expansion of  $(2+x)(3-2x)^7$ .

First, find the first four terms of the binomial expansion of 
$$(3-2x)^7$$

$$= 3^{7} + {7 \choose 1} 3^{6} (-2x) + {7 \choose 2} 3^{5} (-2x)^{2} + {7 \choose 3} 3^{4} (-2x)^{3} + \cdots$$

$$= 2187 - 10206x + 20412x^{2} - 22680x^{3} + \cdots$$

$$\Rightarrow (2+x)(2187 - 10206x + 20412x^{2} - 22680x^{3} + \cdots)$$
Now expand the brackets  $(2+x)(3-2x)^{7}$ 

$$x^3 \text{ term} = 2 \times (-22680x^3) + x \times 20412x^2$$

 $= -24948x^3$ 

The coefficient of  $x^3$  in the binomial expansion of  $(2 + x)(3 - 2x)^7$  is -24948.

There are 2 ways of making the  $x^3$  term: (constant term  $\times x^3$  term) and (x term  $\times x^2$  term)

### **Binomial Estimation**

If the value of x is less than 1, then  $x^n$  gets smaller as n gets larger. If x is small you can sometimes ignore large powers of x to approximate a function or estimate a value.

a. Find the first four terms of the binomial expansion, in ascending powers of x, of

$$\left(1 - \frac{x}{4}\right)^{10} = 1^{10} + {10 \choose 1} 1^9 \left(-\frac{x}{4}\right) + {10 \choose 2} 1^8 \left(-\frac{x}{4}\right)^2 + {10 \choose 3} 1^7 \left(-\frac{x}{4}\right)^3 + \cdots$$
$$= 1 - 2.5x + 2.8125x^2 - 1.875x^3 + \cdots$$

Use your expansion to estimate the value of  $0.975^{10}$ , giving your answer to 4

We want 
$$\left(1 - \frac{x}{4}\right) = 0.975$$

$$\frac{x}{4} = 0.025$$

$$x = 0.1$$
Calculate value of x

Substitute x = 0.1 into the expansion for  $\left(1 - \frac{x}{4}\right)^{10}$  from part a:  $0.975^{10} \approx 1 - 0.25 + 0.028125 - 0.001875$ 

 $0.975^{10} \approx 0.7763 \text{ to 4 d.p}$ 

Using a calculator,  $0.975^{10} = 0.77632962$ . so, approximation is correct.





