1. Prove by contradiction that there is no greatest even positive integer. **[3]**

2. The first term in an arithmetic series is $(5t + 3)$, where t is a positive integer. The last term is $(17t + 11)$ and the common difference is 4. Show that the sum of the series is divisible by 12 when, and only when, t is odd. **[7]**

3. Prove algebraically that $n^3 + 3n - 1$ is odd for all positive integers n. [4]

4. It is given that n is an integer. Prove by contradiction the following statement.

n^2 is even $\Rightarrow$ n is even [5]

5. Charlie claims to have proved the following statement.

"The sum of a square number and a prime number cannot be a square number."

(a) Give an example to show that Charlie's statement is not true. [1]

Charlie's attempt at a proof is below.

Assume that the statement is not true.
⇒ There exist integers n and m and a prime p such that $n^2 + p = m^2$.
⇒ $p = m^2 - n^2$
⇒ $p = (m - n)(m + n)$
⇒ p is the product of two integers.
⇒ p is not prime, which is a contradiction.
⇒ Charlie's statement is true.

[1]

(b) Explain the error that Charlie has made.

(c) Given that 853 is a prime number, find the square number S such that $S + 853$ is also a square number. [4]

6.

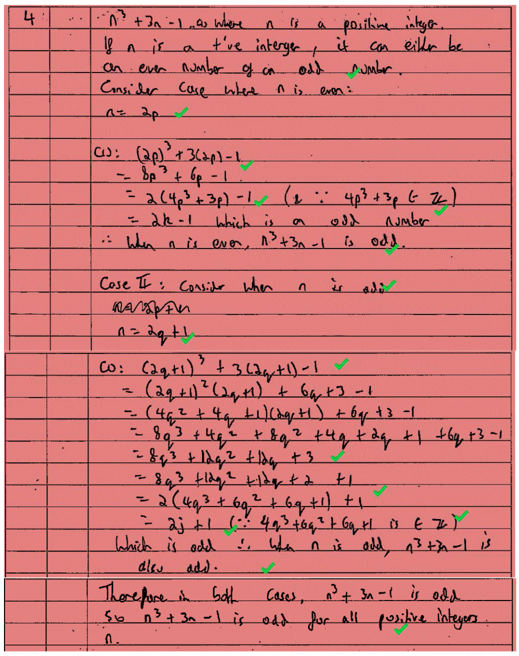
Prove that the sum of the squares of any two consecutive integers is of the form $4k + 1$, where k is an integer.

[4]

END OF QUESTION paper

Question			Answer/Indicative content	Marks	Guidance	
1			Assume that there is a greatest even positive integer $N = 2k$	*E1(AO2.1)	Proof must start with an assumption for contradiction	
			$N + 2 = 2k + 2 = 2(k + 1)$	M1(AO2.1)		
			Which is even and $N + 2 > N$ This contradicts the assumption Therefore there can be no greatest even positive integer	dep*E1(AO2.4) [3]	There must be a statement denying the assumption for the final **E1**	
			Total	3		
2			$(5t + 3) + 4(n - 1) = (17t + 11)$	M1(AO3.1a)	Attempt to use $a + (n - 1)d = l$	
			$n = 3t + 3$	A1(AO2.1)	Obtain $n = 3t + 3$	
			$S_N = \frac{1}{2}(3t + 3)\{(5t + 3) + (17t + 11)\}$ $S_N = \frac{1}{2}(3t + 3)(22t + 14) = 3(t + 1)(11t + 7)$	M1(AO2.1) A1(AO2.1)	Attempt to find sum of AP Obtain $S_N = 3(t + 1)(11t + 7)$ oe	
			When t is odd, $t = 2k + 1$ so	E1(AO2.2a)	Consider S_N when t is odd	Allow consideration of odd and even factors
			$S_N = 3(2k + 2)(22t + 18)$			
			$= 12(k + 1)(11k + 9)$ hence multiple of 12	E1(AO2.4)	Fully correct and convincing proof	
			When t is even, $t = 2k$ so			
			$S_N = 3(2k + 1)(22k + 7)$ hence always odd	E1(AO2.4) [7]	Allow worded eg 3 × odd × odd	
			Total	7		

Question			Answer/Indicative content	Marks	Guidance	
3			If n is even then n can be written as $2m$. $n^3 + 3n - 1 = 8m^3 + 6m - 1$	E1 (AO 2.1)	Consider when n is even	Substitute $2m$ or equiv Must include reasoning, including that $2m$ represents an even number
			$= 2(4m^3 + 3m) - 1$ For all m, $2(4m^3 + 3m)$ is even, hence $2(4m^3 + 3m) - 1$ is odd	E1 (AO 2.4)	Conclude from useable form	Must be of a form where odd can be easily deduced **SR E1** for If n is even, n^3 is even, $3n$ is even, hence $n^3 + 3n$ is even + even = even and therefore $n^3 + 3n - 1$ is even – odd = odd Each step must be justified
			If n is odd then n can be written as $2m + 1$ $n^3 + 3n - 1 = 8m^3 + 12m^2 + 6m + 1 + 6m + 3 - 1$ $= 8m^3 + 12m^2 + 12m + 3$	E1 (AO 2.1)	Consider when n is odd	Substitute $2m + 1$ or equiv Must include reasoning, including that $2m + 1$ represents an odd number
			$= 2(4m^3 + 6m^2 + 6m) + 3$ For all m, $2(4m^3 + 6m^2 + 6m)$ is even, hence $2(4m^3 + 6m^2 + 6m) + 3$ is odd	E1 (AO 2.4) [4]	Conclude from useable form	Must be of a form where odd can be easily deduced **SR E1** for If n is odd, n^3 is odd, $3n$ is odd, hence $n^3 + 3n$ is odd + odd = even and therefore $n^3 + 3n - 1$ is even – odd = odd Each step must be justified

Question			Answer/Indicative content	Marks	Guidance
					<u>Examiner's Comments</u> Candidates were expected to provide an algebraic proof, considering both odd and even *n* and justifying their method. Many candidates made it clear which case was being considered, whereas others introduced 2*k* and 2*k* + 1, but with no explanation of what these terms represented. The algebraic manipulation was usually correct and candidates then wrote the expressions in a useable form. Some candidates then simply stated that the expression must be odd, but did not justify this statement. A number of candidates provided a worded argument based on the use of odd and even numbers. As long as each step was explained then partial credit was given, but full credit in this question was only available for convincing use of algebra. Exemplar 2  4 n^3+3n-1, where n is a positive integer. If n is a +'ve interger, it can either be an even number or an odd number. Consider case where n is even: $n = 2p$ (i): $(2p)^3+3(2p)-1$ $= 8p^3+6p-1$ $= 2(4p^3+3p)-1$ (k ∵ $4p^3+3p \in \mathbb{Z}$) $= 2k-1$ which is an odd number ∴ When n is even, n^3+3n-1 is odd. Case II: Consider when n is odd ~~$n = 2p+1$~~ $n = 2q+1$ (ii): $(2q+1)^3+3(2q+1)-1$ $= (2q+1)^2(2q+1)+6q+3-1$ $= (4q^2+4q+1)(2q+1)+6q+3-1$ $= 8q^3+4q^2+8q^2+4q+2q+1+6q+3-1$ $= 8q^3+12q^2+12q+3$ $= 8q^3+12q^2+12q+2+1$ $= 2(4q^3+6q^2+6q+1)+1$ $= 2j+1$ (∵ $4q^3+6q^2+6q+1$ is $\in \mathbb{Z}$) Which is odd ∴ When n is odd, n^3+3n-1 is also odd. Therefore in both cases, n^3+3n-1 is odd So n^3+3n-1 is odd for all positive integers n. This candidate states that they are using 2*p* to represent an even number; simply using 2*p* with no justification would not have gained the first mark. Having

Question			Answer/Indicative content	Marks	Guidance
					substituted and rearranged they then clearly explain that their expression is of the form $2k - 1$, which represents an odd number. An alternative explanation would be to say that for all p, $2(4p^3 + 3p)$ is even as it has a factor of 2 hence $2(4p^3 + 3p) - 1$ is odd. Simply concluding 'odd' but with no justification would not have gained the second mark. This candidate's proof of why the expression is odd when n is odd is equally convincing, so this response gained full credit.
			Total	4	
4			n^2 is even.		
			Assume n is odd, ie $n = 2r + 1$, where r is an int.	M1(AO 3.1a)	
			$n^2 = 4r^2 + 4r + 1$	A1(AO 1.1)	
			$= 4(r^2 + r) + 1$	M1(AO 2.1)	
			Hence n^2 is odd.	E1(AO 2.1)	Stated
			This contradicts the original statement, hence assumption is false, hence n is even	E1(AO 2.2a)	All three phrases essential
				[5]	
			Total	5	

Question			Answer/Indicative content	Marks	Guidance
5		a	eg 1 + 3 = 4 or 4 + 5 = 9 or 9 + 7 = 16	B1 (AO 1.1) [1]	or 25 + 11 = 36 or any correct example **Examiner's Comments** Almost all candidates answered this correctly
		b	If $m - n = 1$ (or –1) then $(m - n)(m + n)$ could be prime	E1 (AO 2.3) [1]	or One of the factors of p could be 1 (or if $m + n = 1$) **Examiner's Comments** Many candidates identified the error correctly although a large variety of incorrect suggestions were seen, such as "He has proved something that is false is correct, but that doesn't prove his point correct.", "He assumes that the statement is not true.", "p could be 0", "$m + n$ could be zero.", "He hasn't chosen m and n to be integers.", "p is not the product of two integers, $p = m^2 - n^2$.", "He has not stated that m and n are square numbers.".

Question			Answer/Indicative content	Marks	Guidance	
		c	Let $S = n^2$			$853 = m^2 - n^2$ & $m - n = 1$
			$\Rightarrow$ Other square number is $(n + 1)^2$	M1 (AO 3.1a)	or Other square number is $(\sqrt{S} + 1)^2$	$\Rightarrow 853 = m + n$
			$\Rightarrow 853 = (n + 1)^2 - n^2 = 2n + 1$	M1 (AO 2.2a)	$\Rightarrow 853 = (\sqrt{S} + 1)^2 - S = 2\sqrt{S} + 1$	$\Rightarrow 853 = 2n + 1$
			$\Rightarrow n = 426$	A1 (AO 1.1)	$\Rightarrow \sqrt{S} = 426$	$\Rightarrow n = 426$
			$\Rightarrow S = 181476$	A1 (AO 3.2a)	$\Rightarrow S = 181476$	$\Rightarrow S = 181476$
					$m - n = 1$, $m + n = 853$ M1 $2m = 854$ M1 $m = 427$ $n = 426$ A1 $n^2 = 181476$ A1	T & I: 426 seen M1M1A1 $S = 181476$ A1
				[4]	**Examiner's Comments** Some candidates recognised that the starting point was $m - n = 1$. Most of these proceeded to obtain the correct answer (although a few squared 427 instead of 426). Many candidates, however, did not appreciate the link with part (b) and attempted trial and improvement, without success.	
			Total	6		
6			$n^2 + (n + 1)^2 = 2n^2 + 2n + 1$	M1 (AO3.1a)	Attempted	
			$= 2n(n + 1) + 1$	M1 (AO2.1)	This form	
			Either n or $n + 1$ is even			
			$\Rightarrow 2n(n + 1)$ is a multiple of 4 (or is of form $4k$)	A1 (AO2.4)	Statement including reason	
			$\Rightarrow n^2 + (n + 1)^2$ is of the form $4k + 1$	B1 (AO2.4) [4]	Statement of result, dependent on correct working	
			Total	4		