PROOF — Answers

1 **a** e.g. $a = -2, b = 1 \Rightarrow a^2 - b^2 = 4 - 1 = 3 \Rightarrow a^2 - b^2 > 0$
and $a - b = -2 - 1 = -3 \Rightarrow a - b < 0$
[any negative value of a such that $|a| > |b|$]

b 7 — 7 is prime and divisible by 7 [no other examples]

c e.g. $x = \sqrt{2}, y = 2\sqrt{2} \Rightarrow x$ and y irrational
and $xy = 4$ which is rational [many other examples]

d e.g. $x = -90 \Rightarrow \cos(90 - |x|)^\circ = \cos 0 = 1$
and $\sin x^\circ = \sin(-90^\circ) = -1$ [any –ve x except multiples of 180]

2 **a** true — any number divisible by 6 is also divisible by 2 and $\therefore$ not prime

b

n	1	2	3	4	5
$3^n + 2$	5	11	29	83	245

false e.g. $n = 5 \Rightarrow 3^n + 2 = 245$ which is divisible by 5 and $\therefore$ not prime
[many other examples]

c false e.g. $n = 4 \Rightarrow \sqrt{n} = 2$ which is rational [many other examples]

d true — b divisible by $c \Rightarrow b = kc,\ k \in \mathbb{Z}$
a divisible by $b \Rightarrow a = lb,\ l \in \mathbb{Z} \Rightarrow a = klc \ \therefore a$ is divisible by c

3 **a** assume n^3 odd and n even, where $n \in \mathbb{Z}^+$
n even $\Rightarrow n = 2m,\ m \in \mathbb{Z}$
$\Rightarrow n^3 = (2m)^3 = 8m^3 = 2(4m^3)$
$4m^3 \in \mathbb{Z} \ \therefore n^3$ even
$\Rightarrow$ contradiction $\therefore n$ odd

b assume x irrational and $\sqrt{x}$ rational
$\sqrt{x}$ rational $\Rightarrow \sqrt{x} = \frac{p}{q},\ p, q \in \mathbb{Z}$
$\Rightarrow x = \frac{p^2}{q^2},\ p^2, q^2 \in \mathbb{Z} \ \therefore x$ rational
$\Rightarrow$ contradiction $\therefore \sqrt{x}$ irrational

c assume bc not divisible by a and b divisible by a where $a, b, c \in \mathbb{Z}$
b divisible by $a \Rightarrow b = ka,\ k \in \mathbb{Z}$
$\Rightarrow bc = kac$ which is divisible by a
$\Rightarrow$ contradiction $\therefore b$ is not divisible by a

d assume $n^2 - 4n$ odd and n even, where $n \in \mathbb{Z}^+$
n even $\Rightarrow n = 2m,\ m \in \mathbb{Z}$
$\Rightarrow n^2 - 4n = (2m)^2 - 4(2m) = 4m^2 - 8m = 2(2m^2 - 4m)$
$2m^2 - 4m \in \mathbb{Z} \ \therefore n^2 - 4n$ even
$\Rightarrow$ contradiction $\therefore n$ odd

e assume $m^2 - n^2 = 6$, where $m, n \in \mathbb{Z}^+$
$m^2 - n^2 = 6 \Rightarrow (m + n)(m - n) = 6$
$m, n \in \mathbb{Z}^+ \Rightarrow (m + n), (m - n) \in \mathbb{Z},\ (m + n) > (m - n)$ and $(m + n) > 0$
$\therefore m + n = 6$ and $m - n = 1$ or $m + n = 3$ and $m - n = 2$
adding $\Rightarrow 2m = 7$ or $2m = 5$
$\Rightarrow m = \frac{7}{2}$ or $m = \frac{5}{2} \Rightarrow m$ not an integer
$\Rightarrow$ contradiction $\therefore$ no positive integer solutions

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4 **a** assume $x^2 + y^2$ divisible by 4 and x, y odd integers

x, y odd $\Rightarrow$ $x = 2m + 1,\ m \in \mathbb{Z}$ and $y = 2n + 1,\ n \in \mathbb{Z}$

$\Rightarrow$ $x^2 + y^2 = (2m + 1)^2 + (2n + 1)^2$

$= 4m^2 + 4m + 1 + 4n^2 + 4n + 1$

$= 4(m^2 + m + n^2 + n) + 2$

$m^2 + m + n^2 + n \in \mathbb{Z}$ $\therefore$ $x^2 + y^2$ not divisible by 4

$\Rightarrow$ contradiction $\therefore$ x and y not both odd

b assume $x^2 + y^2$ divisible by 4, x odd integer and y even integer

x odd, y even $\Rightarrow$ $x = 2m + 1,\ m \in \mathbb{Z}$ and $y = 2n,\ n \in \mathbb{Z}$

$\Rightarrow$ $x^2 + y^2 = (2m + 1)^2 + (2n)^2$

$= 4m^2 + 4m + 1 + 4n^2$

$= 4(m^2 + m + n^2) + 1$

$m^2 + m + n^2 \in \mathbb{Z}$ $\therefore$ $x^2 + y^2$ not divisible by 4

$\Rightarrow$ contradiction $\therefore$ x odd and y even not possible

same argument applies with x even and y odd

part **a** shows x and y can't both be odd

$\therefore$ x and y both even

5 **a** false e.g. $a = 2, b = 4$ $\Rightarrow$ $\log_a b = 2$ which is rational

[many other examples]

b true $(2n + 1)$ and $(2n + 3)$, $n \in \mathbb{Z}$ represent any two consecutive odd integers

$(2n + 3)^2 - (2n + 1)^2 = 4n^2 + 12n + 9 - (4n^2 + 4n + 1)$

$= 8n + 8$

$= 8(n + 1)$

$n + 1 \in \mathbb{Z}$ $\therefore$ difference is divisible by 8

c false e.g. $n = 13$ $\Rightarrow$ $n^2 + 3n + 13 = 13(13 + 3 + 1)$ which is divisible by 13

[many other examples]

d true $x^2 - 2y(x - y) = x^2 - 2xy + 2y^2$

$= x^2 - 2xy + y^2 + y^2$

$= (x - y)^2 + y^2$

for real x and y, $(x - y)^2 \geq 0$ and $y^2 \geq 0$ $\therefore$ $x^2 - 2y(x - y) \geq 0$

6 **a** $\sqrt{2} = \frac{p}{q}$, $p, q \in \mathbb{Z}$ $\Rightarrow$ $2 = \frac{p^2}{q^2}$ $\Rightarrow$ $p^2 = 2q^2$

$\Rightarrow$ p^2 even $\Rightarrow$ p even

b assume $\sqrt{2}$ rational $\Rightarrow$ $\sqrt{2} = \frac{p}{q}$, $p, q \in \mathbb{Z}$ and p, q co-prime

part **a** $\Rightarrow$ p even $\Rightarrow$ $p = 2n,\ n \in \mathbb{Z}$

$\Rightarrow$ $(2n)^2 = 2q^2$

$\Rightarrow$ $q^2 = 2n^2$

$\Rightarrow$ q^2 even $\Rightarrow$ q even

$\Rightarrow$ p and q both even $\therefore$ not co-prime

$\Rightarrow$ contradiction $\therefore$ $\sqrt{2}$ is irrational

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