1. The smallest of three consecutive positive integers is n. Find the difference between the squares of the smallest and largest of these three integers, and hence prove that this difference is four times the middle one of these three integers. [4]

2. P and Q are consecutive **odd** positive integers such that $P > Q$.

Prove that $P^2 - Q^2$ is a multiple of 8. [3]

3. You are given that n, $n + 1$ and $n + 2$ are three consecutive integers.

i. Expand and simplify $n^2 + (n + 1)^2 + (n + 2)^2$.

[2]

ii. For what values of n will the sum of the squares of these three consecutive integers be an even number?
Give a reason for your answer.

[2]

4. i. Disprove the following statement:

$$3^n + 2 \text{ is prime for all integers } n \geqslant 0.$$

[2]

ii. Prove that no number of the form 3^n (where n is a positive integer) has 5 as its final digit.

[2]

5. i. Factorise fully $n^3 - n$.

[2]

ii. Hence prove that, if n is an integer, $n^3 - n$ is divisible by 6.

[2]

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6. You are given that n is a positive integer.

By expressing $x^{2n} - 1$ as a product of two factors, prove that $2^{2n} - 1$ is divisible by 3.

[4]

7. By finding a counter example, disprove the following statement.

If p and q are non-zero real numbers with $p < q$, then $\frac{1}{p} > \frac{1}{q}$. [2]

8. The spreadsheet in Fig. 5 shows a multiplication table. The numbers 35, 36, 35 in the shaded cells are of the form $n^2 - 1$, n^2, $n^2 - 1$, where $n = 6$. This pattern can also be seen for the other square numbers on the diagonal of the table in cells B2, C3, …, I9.

	A	B	C	D	E	F	G	H	I	J
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Fig. 5

The spreadsheet can be extended to include larger numbers. Prove that the pattern holds for all integers
$n > 1$. [3]

9. Use a counter example to disprove the following statement.

$$2^n - 1 \text{ is prime for all } n > 1$$

[2]

10. You are given that the sum of the interior angles of a polygon with n sides is $180(n - 2)°$. Using this result, or otherwise, prove that the interior angle of a regular polygon cannot be 155°. [3]

END OF QUESTION paper

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Mark scheme

Question	Answer/Indicative content	Marks	Part marks and guidance	
1	$n \quad n+1 \quad n+2$ soi	B1	may be earned later allow ft for next three marks for other general consecutive integers eg $n-1 \quad n \quad n+1$	
	$(n+2)^2 - n^2$ soi	M1	for other integers in terms of n (eg $2n$, $2n+1$, $2n+2$ or $2n+1$, $2n+3$, $2n+5$) allow ft for this M1 only	allow $n^2 - (n+2)^2$ for M1 then A0 for negative answer; may still earn last B1
	$4n+4$ obtained with at least one interim step shown	A1		
	$4(n+1)$ or $\dfrac{4n+4}{4} = n+1$	B1	may be obtained independently	B0 for $n + 1 \times 4$
		[4]		

Examiner's Comments

The majority of candidates that attempted this standard proof question gained full marks, showing the needed interim step(s) to obtain the corresponding accuracy marks. A minority chose wrong expressions for the three integers (e.g. n, $2n$, $3n$). Unfortunately candidates missing the middle term of $4n$ when squaring the $(n+2)$ term was seen quite often. Some candidates considered the first term squared minus the last term squared and then conveniently ignored the negative signs. A handful of candidates attempted an entirely numerical approach.

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			Total	4		
2			$(2n+1)^2 - (2n-1)^2$ oe $4n^2 + 4n + 1 - (4n^2 - 4n + 1)$ $= 8n$ (so multiple of 8)	B1(AO2.1) M1(AO1.1) A1(AO2.4) [3]	Allow one slip eg sign error Note: Numerical verification 0 OR $P = Q + 2$ $P^2 - Q^2 = (Q+2)^2 - Q^2$ B1 $= 4(Q+1)$ (actorized) M1 $(Q+1)$ divisible by 2 so $4(Q+1)$ is multiple of 8 A1 OR $Q = P - 2$ **Examiner's Comments** There were quite a number of completely correct proofs, usually starting with P = 2*n* + 1 or 2*n* + 3. A small number of candidates put P = 2*n* + 1 and Q = 2*n* + 3; these candidates could gain some credit for subsequent work. A considerable number of candidates did not use the fact that *P* and *Q* were odd at the start of the proof, and worked with *P* and (*P* – 2) or *Q* and (*Q* + 2), or with *n* and (*n* + 2) or (*n* – 1) and (*n* + 1). These candidates generally showed that $P^2 - Q^2$ is a multiple of 4; full credit could have been awarded by using the fact that *P* and *Q* are odd at this point to prove the required result, but this was not seen. Attempts to show this by giving numerical responses were also seen, but these scored 0. Occasionally a candidate gave several numerical responses, and claimed that this was 'proof by exhaustion'.	
			Total	3		
3		i	$3n^2 + 6n + 5$ isw	B2	M1 for a correct expansion of at least one of $(n+1)^2$ and $(n+2)^2$	
		ii	odd numbers with valid explanation	B2	marks dep on 9(i) correct or starting again for B2 must see at least odd × odd = odd	accept a full valid argument using odd and even from starting again

					[for $3n^2$] (or when n is odd, $[3]n^2$ is odd) and odd [+ even] + odd = even soi condone lack of odd × even = even for $6n$; condone no consideration of n being even or B2 for deductive argument such as: $6n$ is always even [and 5 is odd] so $3n^2$ must be odd so n is odd B1 for odd numbers with a correct partial explanation or a partially correct explanation or B1 for an otherwise fully correct argument for odd numbers but with conclusion positive odd numbers or conclusion negative odd numbers B0 for just a few trials and conclusion **Examiner's Comments** The straightforward algebra in the first part was done correctly by most candidates. The most common errors were to write $(n + 1)^2$ as $n^2 + 1$, or sometimes $n^2 + 2n + 2$ and $(n + 2)^2$ as $n^2 + 4$. There was some encouraging work in the proof part with a number of slightly different methods being demonstrated. The majority considered the three terms that they had found in (i) but others went further and expressed the quadratic function as $3n(n + 2) + 5$ or $3(n + 1)^2 + 2$. As these were the more capable candidates they were then often argued the case elegantly. Some candidates returned to the original function successfully with a few replacing n by $2m + 1$ and expanding to find a factor of 2. The candidates who fared badly were those who failed to draw any conclusion at all, those who attempted to use an incorrect expression from (i) or those who just tried to show it with some numerical values.	ignore numerical trials or examples in this part – only a generalised argument can gain credit
			Total	4		
4		i	$3^5 + 2 = 245$ [which is not prime]	M1	Attempt to find counter-example	If A0, allow M1 for $3^n + 2$ correctly
		i		A1	correct counter-example identified **Examiner's Comments**	evaluated for 3 values of n

					This proved to be very straightforward, with nearly everyone quoting the first correct counter-example of 245 (though a few came up with some much larger numbers).	
		ii	$(3^0 = 1)$, $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, …	M1	Evaluate 3^n for $n = 0$ to 4 or 1 to 5	allow just final digit written
		ii	so units digits cycle through 1, 3, 9, 7, 1, 3,			
		ii	…	A1		
		ii	so cannot be a '5'. **OR**			
		ii	$3n$ is not divisible by 5	B1		
		ii	all numbers ending in '5' are divisible by 5. so its last digit cannot be a '5'	B1	must state conclusion for B2 **Examiner's Comments** This was not quite so easy as part (i). Most candidates who got full marks spotted the cyclic pattern in the units digits of 3^n as n increases. However, a significant minority evaluated 3^n for $n = 0$ to 9 and then cited 'proof by exhaustion'. The second approach, less commonly used, was to use the fact that numbers ending in '5' must be multiples of 5, and 3^n contains no factors of 5. However, many candidates who used this approach were unable to express the argument clearly enough and made incorrect statements.	
			Total	4		
5		i	$n^3 - n = n(n^2 - 1)$	B1	two correct factors **Examiner's Comments** Many candidates failed to factorise the $n^2 - 1$, leaving their answer as $n(n^2 - 1)$. This rendered the second part of the question very difficult.	
		i	$= n(n - 1)(n + 1)$	B1		
		ii	$n - 1$, n and $n + 1$ are consecutive integers	B1		
		ii	so at least one is even, and one is div by 3 [$\Rightarrow n^3 - n$ is div by 6]	B1	**Examiner's Comments** There were two ideas needed here, the realisation that $n - 1$, n and $n + 1$ were consecutive integers, and that the product contained factors 2 and 3. Many candidates argued that the product had to be even, but this was not enough to gain credit. Others, predictably, verified the result with a few	

					values of n, often describing this as 'proof by exhaustion'.	
			Total	4		
6			$x^{2n} - 1 = (x^n - 1)(x^n + 1)$	B1		
			one of $2^n - 1$, $2^n + 1$ is divisible by three $2^n - 1$, 2^n and $2^n + 1$ are consecutive integers;	M1		award notwithstanding false reasoning condone 'factor' for 'multiple'
			one must therefore be divisible by 3;	A1		
			but 2^n is not, so one of the other two is	A1	if justified, correct reason must be given	
			2^n is not div by 3, and so has remainder 1 or 2 when divided by 3; if remainder is 1, $2^n - 1$ is div by 3; if remainder is 2, then $2^n + 1$ is div by 3 [so $2^{2n} - 1$ is divisible by 3]	A2		**Examiner's Comments** The first B1 for factorising $x^{2n} - 1$ was well done, but convincing proofs of the divisibility of $2^{2n} - 1$ by 3 were few and far between. We awarded M1 if candidates recognised that either $2^n - 1$ or $2^n + 1$ were divisible by 3, and two 'A' marks for proving this. The next 'A' mark was gained for stating that the consecutive numbers 2^n-1, 2^n and 2^{n+1} must include a multiple of 3, and the final mark for stating that 2^n is **not** divisible by 3; however, many candidates wrongly stated that 2^n was even and therefore not divisible by 3, or that two consecutive odd numbers must include a multiple of 3. The most elegant alternative solution seen was: $x^{2n} - 1 = (x^2 - 1)(x^{2n-2} + x^{2n-4} + \ldots + 1) \Rightarrow 2^{2n} - 1 = (2^2 - 1)(2^{2n-2} + 2^{2n-4} + \ldots + 1) = 3m$, where m is an integer. The language used by candidates in their explanations was often rather imprecise. In particular, the terms 'factor' and 'multiple' were often used incorrectly.
			Total	4		

7			E.g. $p = -1$, $q = 2$ $\frac{1}{p} = -1$, $\frac{1}{q} = \frac{1}{2}$ So $\frac{1}{p} < \frac{1}{q}$ for these values.	B1(AO3.1a) E1(AO2.1) [2]	correct counter example stated shown	
			Total	2		
8			Use of $n + 1$ and $n - 1$ $(n + 1)(n - 1) = n^2 - 1$ Therefore (the conjecture is) true for all positive integers n (greater than 1)	B1(AO 2.1) M1(AO 1.1a) E1(AO 2.1) [3]	For both expressions seen Must be multiplied with attempt to expand brackets Clear conclusion must be stated	
			Total	3		
9			2^k evaluated for any positive integer k eg $2^4 - 1 = 15 = 5 \times 3$ which is not prime	M1(AO 1.1) A1(AO 2.4) [2]	Any positive integer which generates any non-prime	
			Total	2		
10			Suppose the polygon has n sides. Then $180(n - 2) = 155n$ $\Rightarrow 25n = 360$ [$\Rightarrow n = 14.4$]	M1 A1	or sum of ext	

			which is impossible as n is an integer So no regular polygon has interior angle 155° **or** When n = 14, int angle = 180×12/14 = 154.29° When n = 15, int angle = 180×13/15 = 156° So no n which gives an interior angle 155°.	A1cao B1 B1 B1 [3]	angles = 360° so 25n = 360 or 72/5 clear statement of conclusion accept 154° **Examiner's Comments** Candidates scored full marks or zero marks in roughly equal numbers here. Most gave the first method shown in the mark scheme, namely solving 180(n – 2) = 155n to get n = 14.4, but we also saw some examples of the second approach, finding the interior angles for 14 and 15 sides. By far the most common error was to solve 180(n – 2) = 155, getting n = 2.86.	
			Total	3		