1. (i) Show that $x^2 - 8x + 17 > 0$ for all real values of x

(3)

(ii) "If I add 3 to a number and square the sum, the result is greater than the square of the original number."

State, giving a reason, if the above statement is always true, sometimes true or never true.

(2)

i) $x^2 - 8x + 17 = (x-4)^2 + 17 - 16$

$= (x-4)^2 + 1$

$(x-4)^2$ is always greater than 0

$\therefore x^2 - 8x + 17 > 0$ for all real values of x

ii) for $n=1$, $(1+3)^2 = 16$ $1^2 = 1$

$16 > 1$ $\therefore$ true for $n=1$

for $n=-3$ $(-3+3)^2 = 0$ $(-3)^2 = 9$

$0 < 9$ $\therefore$ false for $n=-3$

$\therefore$ statement is sometimes true

2. (a) Prove that for all positive values of x and y

$$\sqrt{xy} \leqslant \frac{x+y}{2}$$

(2)

(b) Prove by counter example that this is not true when x and y are both negative.

(1)

(a) Since x and y are both positive, their square roots are real, and so we can use:

$$(\sqrt{x} - \sqrt{y})^2 \geq 0$$

$$(\sqrt{x} - \sqrt{y})(\sqrt{x} - \sqrt{y}) \geq 0$$

$$x - 2\sqrt{x}\sqrt{y} + y \geq 0$$

$$x - 2\sqrt{x}\sqrt{y} + y \geq 0$$

$$x + y \geq 2\sqrt{x}\sqrt{y}$$

$$2\sqrt{x}\sqrt{y} \leq x + y$$

$$\therefore \sqrt{x}\sqrt{y} \leq \frac{x+y}{2}$$

$$\therefore \boxed{\sqrt{xy} \leq \frac{x+y}{2}}$$

(b) If $x=-2$ and $y=-3$, then:

$$LHS = \sqrt{(-2)(-3)} = \sqrt{6}$$

$$RHS = \frac{-2+(-3)}{2} = \frac{-2-3}{2} = \frac{-5}{2}$$

so $\sqrt{xy} > \frac{x+y}{2}$ in this case

3. Given $n \in \mathbb{N}$, prove that $n^3 + 2$ is not divisible by 8

(4)

If n is even, then

let $n = 2K$.

$$\frac{n^3+2}{8} = \frac{(2K)^3+2}{8}$$

$$= \frac{8K^3+2}{8}$$

$$= K^3 + \frac{1}{4}$$

This is evidently not a whole number and hence not divisible by 8.

If n is odd, then

let $n = 2K+1$

$$\frac{n^3+2}{8} = \frac{(2K+1)^3+2}{8}$$

$$= \frac{(2K+1)(4K^2+4K+1)+2}{8}$$

DO NOT WRITE IN THIS AREA

$$= \frac{8k^3 + 12k^2 + 6k + 3}{8}$$

$$= \frac{2(4k^3 + 6k^2 + 3k) + 3}{8}$$

The numerator is an odd number because we have an even number + 3, hence odd. Hence this is not divisible by 8 either, and thus for $n \in \mathbb{N}$, $n^3 + 2$ is NOT divisible by 8. //

4. (a) Prove that for all positive values of a and b

$$\frac{4a}{b} + \frac{b}{a} \geqslant 4$$

(4)

(b) Prove, by counter example, that this is not true for all values of a and b.

(1)

a) for all real numbers, their value squared is always ≥ 0

$\frac{4a}{b} + \frac{b}{a} = 4 \Rightarrow 4a^2 + b^2 - 4ab = 0 \Rightarrow (2a-b)^2 = 0$

you can 'reverse engineer' to find how to prove statement

starting point for proof

$\therefore (2a-b)^2 \geq 0$

$4a^2 + b^2 - 4ab \geq 0$

as $a, b > 0$, dividing by either doesn't change direction of inequality : $\frac{4a^2}{ab} + \frac{b^2}{ab} \geq \frac{4ab}{ab}$

$\Rightarrow \frac{4a}{b} + \frac{b}{a} \geq 4$

(a) uses $a, b > 0$

b) so counter example must use negative value

e.g. $a = 5, b = -1$: $\frac{4a}{b} + \frac{b}{a} = -20 - \frac{1}{5} < 4$

5. A student is investigating the following statement about natural numbers.

"$n^3 - n$ is a multiple of 4"

(a) Prove, using algebra, that the statement is true for all odd numbers.

(4)

(b) Use a counterexample to show that the statement is not always true.

(1)

(a) let odd numbers be : $2k+1$ 1

Substitute this into the equation : $(2k+1)^3-(2k+1)$

expand this

$= (2k+1)(2k+1)^2 - 2k - 1$ 1

$= (2k+1)(4k^2+4k+1) - 2k - 1$

$= 8k^3 + 8k^2 + 2k + 4k^2 + 4k + 1 - 2k - 1$

$= 8k^3 + 12k^2 + 4k$

$= 4(2k^3 + 3k^2 + k)$ 1

$\therefore$ Hence, $4 \times \ldots$ is a multiple of 4. So, the statement "$n^3 - n$ is a multiple of 4" is true for all odd numbers. 1

(b) Select an even number (e.g. 2)

Substitute 2 into the equation

$= (2)^3 - (2)$

$= 8 - 2$

$= 6$

$\therefore$ 6 is not a multiple of 4, hence, the statement is not always true. 1

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

6. Complete the table below. The first one has been done for you.

For each statement you must state if it is always true, sometimes true or never true, giving a reason in each case.

Statement	Always True	Sometimes True	Never True	Reason
The quadratic equation $ax^2 + bx + c = 0, \quad (a \neq 0)$ has 2 real roots.		✓		It only has 2 real roots when $b^2 - 4ac > 0$. When $b^2 - 4ac = 0$ it has 1 real root and when $b^2 - 4ac < 0$ it has 0 real roots.
(i) When a real value of x is substituted into $x^2 - 6x + 10$ the result is positive. **(2)**	✓			$x^2-6x+10$ $(x-3)^2-9+10$ $(x-3)^2+1$ $(x-3)^2 \geq 0$ $(x-3)^2+1 \geq 1$
(ii) If $ax > b$ then $x > \frac{b}{a}$ **(2)**		✓		If $a>0$ $ax>b$ $x>\frac{b}{a}$ If $a<0$ $ax>b$ $x<\frac{b}{a}$
(iii) The difference between consecutive square numbers is odd. **(2)**	✓			$n^2, (n+1)^2$ $(n+1)^2-n^2$ $n^2+2n+1-n^2$ $2n+1$ $2n+1$ is the expression for odd numbers.

7. Prove by contradiction that there are no positive integers p and q such that

$$4p^2 - q^2 = 25$$

(4)

Proof by Contradiction :
- assume that the first statement is false
- through logical steps, arrive at a conclusion
- deduce that the original statement must be true

=> let us assume that there are positive integers p and q such that $4p^2 - q^2 = 25$.

$\Rightarrow \quad 4p^2 - q^2 = 25$

$\Rightarrow \quad (2p+q)(2p-q) = 25$ ①

25

$1 \times 25 = 25$

$5 \times 5 = 25$

=> Factors are
- 1 and 25
- 5 and 5

=> If true then $2p+q = 5$ and $2p-q = 5$ ①

$\Rightarrow q = 5 - 2p$ and $q = 2p - 5$

$\Rightarrow 5 - 2p = 2p - 5$

$\Rightarrow 4p = 10$ and therefore $p = \underline{2.5}$ — not an integer

$\Rightarrow q = 2(2.5) - 5 \Rightarrow \underline{q = 0}$ ①

OR If true $2p + q = 25$ and $2p - q = 1$

$\Rightarrow q = 25 - 2p$ and $q = 2p - 1$

$\Rightarrow 25 - 2p = 2p - 1$

$\Rightarrow 4p = 26 \Rightarrow p = \underline{6.5}$ — not an integer

$\Rightarrow q = 2(6.5) - 1 = \underline{12}$

OR if true, then $2p + q = 1$ and $2p - q = 25$

$\Rightarrow q = 1 - 2p$ and $q = 2p - 25$

$\Rightarrow 1 - 2p = 2p - 25 \Rightarrow p = 6.5$ — not an integer

$\Rightarrow q = 1 - 2(6.5) = \underline{-12}$ — q is not positive.

=> This is a contradiction as there are no integer solutions, hence there are no positive integers p and q such that $\underline{4p^2 - q^2 = 25}$. ①

8. Use algebra to prove that the square of any natural number is **either** a multiple of 3 **or** one more than a multiple of 3

(4)

• $3k$ • $3k+1$ • $3k+2$ (we can express natural in this form)

$3k$: $(3k)^2 = 9k^2 = 3 \times 3k^2$ which is a multiple of 3.

$3k+1$: $(3k+1)^2 = (3k+1)(3k+1) = 9k^2+6k+1$ (1)

$= 3(3k^2+2k)+1$ which is one more than a multiple of 3. (1)

$3k+2$: $(3k+2)^2 = (3k+2)(3k+2) = 9k^2+12k+4$ (1)

$= 3(3k^2+4k+1)+1$ which is also one more than a multiple of 3.

=> We have shown that the square of any natural number is either a multiple of 3 or one more than a multiple of 3. (1)

9. (i) Use proof by exhaustion to show that for $n \in \mathbb{N}, n \leqslant 4$

$$(n+1)^3 > 3^n$$

(2)

(ii) Given that $m^3 + 5$ is odd, use proof by contradiction to show, using algebra, that m is even.

(4)

(i) $n = 1$: $(1+1)^3 > 3^1$

$2^3 > 3$

$8 > 3$ → statement is correct

$n = 2$: $(2+1)^3 > 3^2$

$3^3 > 3^2$

$27 > 9$ → statement is correct

$n = 3$: $(3+1)^3 > 3^3$

$4^3 > 27$

$64 > 27$ → statement is correct

$n = 4$: $(4+1)^3 > 3^4$

$5^3 > 3^4$

$125 > 81$ → statement is correct 1

∴ Hence, if $n \leq 4$, $n \in \mathbb{N}$, then $(n+1)^3 > 3^n$ 1

1

(b) Let m be odd : $m = 2p + 1$ 1

$m^3 + 5 = (2p+1)^3 + 5$

$= (2p+1)(2p+1)^2 + 5$

$= (2p+1)(4p^2 + 4p + 1) + 5$

$= 8p^3 + 8p^2 + 2p + 4p^2 + 4p + 1 + 5$

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

$= 8p^3 + 12p^2 + 6p + 6$ 1

$= 2(4p^3 + 6p^2 + 3p + 3)$, which is even.

$\therefore$ $2(4p^3 + 6p^2 + 3p + 3)$ is even because it is a multiple of 2. Any multiple of 2 is an even number. Since $m^3 + 5$ is even when m is odd, this is a contradiction to the given statement. Hence, if $m^3 + 5$ is odd, then m must be even. 1