Mathematical Proof Cheat Sheet

Pure Year 1

A proof is a logical and structured argument to show that a mathematical statement (or conjecture) is always true. A statement that has been proven is called a **theorem**. A statement that has yet to be proven is called a **conjecture**. A mathematical proof needs to show that something is true in every case.

Below are a series of steps that you will have to use to prove a mathematical statement:

Known facts of theorem → Clearly shown logical steps → Statement of proof

In a mathematical proof, you must ensure you:

- State any information or any assumptions that you are using
- Show every step of your proof clearly
- Follow every step logically from the previous steps
- Cover all the possible cases
- Write a statement of proof at the end of your working

Proof by deduction:
Sarting from known facts or definitions, then using logical steps to reach the desired conclusion.

Example 1: Proof that $n^2 - n$ is an even number for all values of n.

Start by factorising the term as follow:

$n^2 - n = n(n-1)$

Any number is either ODD or EVEN. Start by assuming n is ODD:
If n is ODD, then that means $n-1$ is EVEN. Hence:

$n \times (n-1) \Longrightarrow$ ODD × EVEN = EVEN

Next, we assume n is EVEN:
If n is EVEN, then that means $n-1$ is ODD. Hence:

$n \times (n-1) \Longrightarrow$ EVEN × ODD= EVEN

Conclusion:
$\therefore n^2 - n$ is even for all values of n

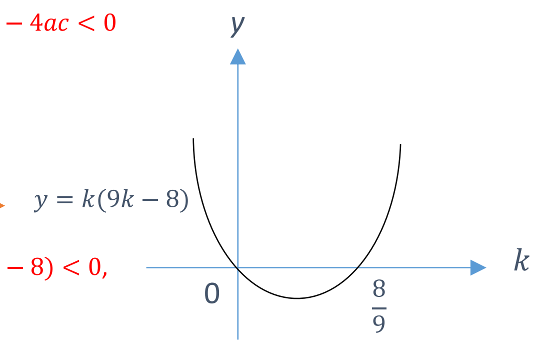
Example 2: The equation $kx^2 + 3kx + 2 = 0$, where k is a constant, has no real roots. Prove that k satisfies the inequality $0 \le k < \frac{8}{9}$

State which assumption or information you are using at each stage of your proof. Start by stating:

$kx^2 + 3kx + 2 = 0$ has no real roots, so $b^2 - 4ac < 0$

$(3k)^2 - 4(k)(2) < 0$
$9k^2 - 8k < 0$
$k(9k-8) < 0$

Sketch the graph of $y = k(9k-8)$ →



From the graph, we can see that when $k(9k-8) < 0$,
$0 < k < \frac{8}{9}$

Meanwhile, when $k = 0$,

$(0)x^2 + 3(0)x + 2 = 0$

$2 = 0$

This is impossible, which means there are no real roots when $k = 0$

Hence, combining $0 < k < \frac{8}{9}$ and $k = 0$ gives $0 \le k < \frac{8}{9}$

In conclusion, k satisfies the inequality $0 \le k < \frac{8}{9}$

To prove an identity, you should:

- Start with the expression on one side of the identity
- Manipulate that expression algebraically until it matches the other side
- Show every step of your algebraic working

It is important to note that the symbol ≡ means 'is always equals to'. It shows that two expressions are mathematically identical.

Example 3: Prove that $(3x+2)(x-5)(x+7) \equiv 3x^3 + 8x^2 - 101x - 70$

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$(3x+2)(x^2 + 2x - 35) \equiv 3x^3 + 8x^2 - 101x - 70$

$3x^3 + 6x^2 - 105x + 2x^2 + 4x - 70 \equiv 3x^3 + 8x^2 - 101x - 70$

$3x^3 + 8x^2 - 101x - 70 \equiv 3x^3 + 8x^2 - 101x - 70$

As the left-hand side is equal to the right-hand side, we have proved the identity.

Example 4: Prove that $(x - \frac{2}{x})^3 \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$

$(x - \frac{2}{x})^3 \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$

$(x - \frac{2}{x})(x - \frac{2}{x})(x - \frac{2}{x}) \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$

$\left(x - \frac{2}{x}\right)(x^2 - 4 + \frac{4}{x^2}) \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$

$x^3 - 4x + \frac{4}{x} - 2x + \frac{8}{x} - \frac{8}{x^3} \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$

$x^3 - 6x + \frac{12}{x} - \frac{8}{x^3} \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$

Methods of proof:
You can prove a mathematical statement using proof by exhaustion. This method requires you to break the statement into smaller cases and prove each case separately.

This method is better suited to a small number of results. You cannot use one example to prove a statement is true as one example is only one case.

Example 5: Prove that all square numbers are either a multiple of 4 or 1 more than a multiple of 4.

First, you will need to consider the two cases, odd and even numbers separately. You can write any odd number in the form of $2n+1$ where n is a positive integer.

Hence, for odd numbers:

$(2n+1)^2 = 4n^2 + 4n + 1$
$= 4n(n+1) + 1$

Since $4n(n+1)$ is a multiple of 4, we have that $4n(n+1) + 1$ is one more than a multiple of 4.

Next, you can write any even numbers in the form $2n$ where n is a positive integer.

Hence, for even numbers:

$(2n)^2 = 4n^2$

$4n^2$ is a multiple of 4

Hence, taking both cases into account, all numbers are either odd or even, so all square numbers are either a multiple of 4 or 1 more than a multiple of 4.

Counter-example:
You can prove a mathematical statement is not true by counter-example. A counter-example is one example that does not work for the given statement. To disprove a statement one counter example is enough.

Example 6: Show, by means of a counter-example, that the following inequality does not hold when p and q are both negative

$p + q > \sqrt{4pq}$

Start by taking negative values for both p and q
$p = -1, q = -2$
$p + q = (-1) + (-2) = -1$ - 2 = -3
$\sqrt{4pq} = \sqrt{4(-1)(-2)} = \sqrt{8}$
But -3 < $\sqrt{8}$, i.e. $p + q < \sqrt{4pq}$
Hence by counter example, we proved the inequality is not true for negative values.

Example 7: Prove that the following statement is not true:

'The sum of two consecutive prime numbers is always even'

2 and 3 are both prime numbers appearing consecutively.

$2 + 3 = 5$

5 is an odd number. Hence, the statement is not true.