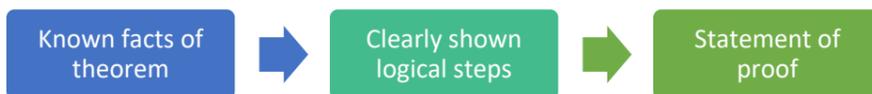


## Mathematical Proof Cheat Sheet

A proof is a logical and structured argument to show that a mathematical statement (or conjecture) is always true. A statement that has been proven is called a **theorem**. A statement that has yet to be proven is called a **conjecture**. A mathematical proof needs to show that something is true in every case.

Below are a series of steps that you will have to use to prove a mathematical statement:



**In a mathematical proof, you must ensure you:**

- State any information or any assumptions that you are using
- Show every step of your proof clearly
- Follow every step logically from the previous steps
- Cover all the possible cases
- Write a statement of proof at the end of your working

**Proof by deduction:**

Sarting from known facts or definitions, then using logical steps to reach the desired conclusion.

**Example 1:** Proof that  $n^2 - n$  is an even number for all values of  $n$ .

Start by factorising the term as follow:

$$n^2 - n = n(n - 1)$$

Any number is either ODD or EVEN. Start by assuming  $n$  is ODD:

If  $n$  is ODD, then that means  $n - 1$  is EVEN. Hence:

$$n \times (n - 1) \Rightarrow \text{ODD} \times \text{EVEN} = \text{EVEN}$$

Next, we assume  $n$  is EVEN:

If  $n$  is EVEN, then that means  $n - 1$  is ODD. Hence:

$$n \times (n - 1) \Rightarrow \text{EVEN} \times \text{ODD} = \text{EVEN}$$

**Conclusion:**

$\therefore n^2 - n$  is even for all values of  $n$

**Example 2:** The equation  $kx^2 + 3kx + 2 = 0$ , where  $k$  is a constant, has no real roots. Prove that  $k$  satisfies the inequality  $0 \leq k < \frac{8}{9}$

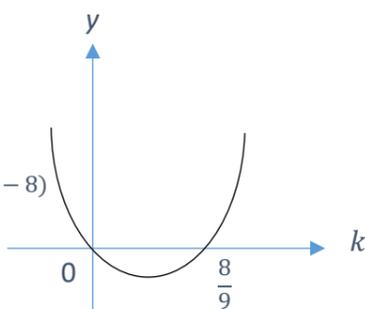
State which assumption or information you are using at each stage of your proof. Start by stating:

$kx^2 + 3kx + 2 = 0$  has no real roots, so  $b^2 - 4ac < 0$

$$\begin{aligned} (3k)^2 - 4(k)(2) &< 0 \\ 9k^2 - 8k &< 0 \\ k(9k - 8) &< 0 \end{aligned}$$

Sketch the graph of  $y = k(9k - 8)$   $\longrightarrow$   $y = k(9k - 8)$

From the graph, we can see that when  $k(9k - 8) < 0$ ,  
 $0 < k < \frac{8}{9}$



Meanwhile, when  $k = 0$ ,

$$(0)x^2 + 3(0)x + 2 = 0$$

$$2 = 0$$

This is impossible, which means there are no real roots when  $k = 0$

Hence, combining  $0 < k < \frac{8}{9}$  and  $k = 0$  gives  $0 \leq k < \frac{8}{9}$

In conclusion,  $k$  satisfies the inequality  $0 \leq k < \frac{8}{9}$

**To prove an identity, you should:**

- Start with the expression on one side of the identity
- Manipulate that expression algebraically until it matches the other side
- Show every step of your algebraic working

It is important to note that the symbol  $\equiv$  means 'is always equals to'. It shows that two expressions are mathematically identical.

**Example 3:** Prove that  $(3x + 2)(x - 5)(x + 7) \equiv 3x^3 + 8x^2 - 101x - 70$

$$(3x + 2)(x - 5)(x + 7) \equiv 3x^3 + 8x^2 - 101x - 70$$

$$(3x + 2)(x^2 + 2x - 35) \equiv 3x^3 + 8x^2 - 101x - 70$$

$$3x^3 + 6x^2 - 105x + 2x^2 + 4x - 70 \equiv 3x^3 + 8x^2 - 101x - 70$$

$$3x^3 + 8x^2 - 101x - 70 \equiv 3x^3 + 8x^2 - 101x - 70$$

As the left-hand side is equal to the right-hand side, we have proved the identity.

**Example 4:** Prove that  $(x - \frac{2}{x})^3 \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$

$$(x - \frac{2}{x})^3 \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$$

$$(x - \frac{2}{x})(x - \frac{2}{x})(x - \frac{2}{x}) \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$$

$$(x - \frac{2}{x})(x^2 - 4 + \frac{4}{x^2}) \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$$

$$x^3 - 4x + \frac{4}{x} - 2x + \frac{8}{x} - \frac{8}{x^3} \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$$

$$x^3 - 6x + \frac{12}{x} - \frac{8}{x^3} \equiv x^3 - 6x + \frac{12}{x} - \frac{8}{x^3}$$

**Methods of proof:**

You can prove a mathematical statement using proof by exhaustion. This method requires you to break the statement into smaller cases and prove each case separately.

This method is better suited to a small number of results. You cannot use one example to prove a statement is true as one example is only one case.

**Example 5:** Prove that all square numbers are either a multiple of 4 or 1 more than a multiple of 4.

First, you will need to consider the two cases, odd and even numbers separately. You can write any odd number in the form of  $2n + 1$  where  $n$  is a positive integer.

Hence, for odd numbers:

$$\begin{aligned} (2n + 1)^2 &= 4n^2 + 4n + 1 \\ &= 4n(n + 1) + 1 \end{aligned}$$

Since  $4n(n + 1)$  is a multiple of 4, we have that  $4n(n + 1) + 1$  is one more than a multiple of 4.

Next, you can write any even numbers in the form  $2n$  where  $n$  is a positive integer.

Hence, for even numbers:

$$(2n)^2 = 4n^2$$

$4n^2$  is a multiple of 4

Hence, taking both cases into account, all numbers are either odd or even, so all square numbers are either a multiple of 4 or 1 more than a multiple of 4.

**Counter-example:**

You can prove a mathematical statement is not true by counter-example. A counter-example is one example that does not work for the given statement. To disprove a statement one counter example is enough.

**Example 6:** Show, by means of a counter-example, that the following inequality does not hold when  $p$  and  $q$  are both negative

$$p + q > \sqrt{4pq}$$

Start by taking negative values for both  $p$  and  $q$

$$\begin{aligned} p &= -1, q = -2 \\ p + q &= (-1) + (-2) = -1 - 2 = -3 \\ \sqrt{4pq} &= \sqrt{4(-1)(-2)} = \sqrt{8} \end{aligned}$$

But  $-3 < \sqrt{8}$ , i.e.  $p + q < \sqrt{4pq}$

Hence by counter example, we proved the inequality is not true for negative values.

**Example 7:** Prove that the following statement is not true:

'The sum of two consecutive prime numbers is always even'

2 and 3 are both prime numbers appearing consecutively.

$$2 + 3 = 5$$

5 is an odd number. Hence, the statement is not true.

