1. The function f(x) is defined by $f(x) = x^4 + x^3 - 2x^2 - 4x - 2$.

(a) Show that $x = -1$ is a root of $f(x) = 0$. [1]

(b) Show that another root of $f(x) = 0$ lies between $x = 1$ and $x = 2$. [2]

(c) Show that $f(x) = (x + 1)g(x)$, where $g(x) = x^3 + ax + b$ and a and b are integers to be determined. [3]

(d) Without further calculation, explain why $g(x) = 0$ has a root between $x = 1$ and $x = 2$. [1]

(e) Use the Newton-Raphson formula to show that an iteration formula for finding roots of $g(x) = 0$ may be written

$$x_{n+1} = \frac{2x_n^{\ 3} + 2}{3x_n^{\ 2} - 2}$$

Determine the root of $g(x) = 0$ which lies between $x = 1$ and $x = 2$ correct to 4 significant figures. [3]

2.

Fig. 15 shows the graph of $f(x) = 2x + \frac{1}{x} + \ln x - 4$.

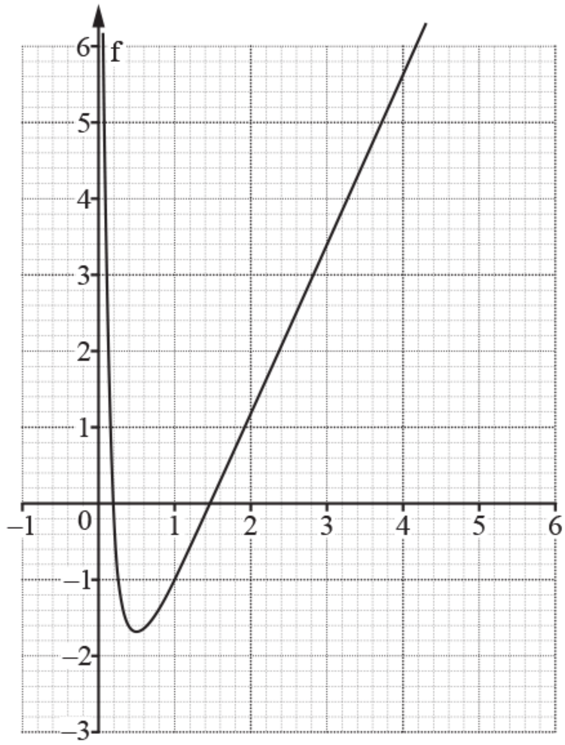


Fig. 15

(a) Show that the equation

$$2x + \frac{1}{x} + \ln x - 4 = 0$$

has a root, α, such that $0.1 < \alpha < 0.9$. [2]

(b) Obtain the following Newton-Raphson iteration for the equation in part **(a)**.

$$x_{r+1} = x_r - \frac{2x_r^3 + x_r + x_r^2(\ln x_r - 4)}{2x_r^2 - 1 + x_r}$$

[3]

(c) Explain why this iteration fails to find α using each of the following starting values.

(i) $x_0 = 0.4$ [2]

(ii) $x_0 = 0.5$ [2]

(iii) $x_0 = 0.6$ [2]

3. Joe uses the Newton-Raphson method to try to solve the equation $x^3 - 3x^2 - 10x + 25 = 0$.

(a) Show that the formula Joe should use is $$x_{n+1} = x_n - \frac{x_n^3 - 3x_n^2 - 10x_n + 25}{3x_n^2 - 6x_n - 10}$$. [1]

(b) Joe uses $x_0 = 4$ in this formula to find a root and obtains the following values:

$x_1 = 3.928\ 571\ 4,$
$x_2 = 3.924\ 992\ 8.$

Joe states that the root must be 3.92 to 2 decimal places and argues that this is because both x_1 and x_2 begin with 3.92.

(i) Comment on the validity of Joe's argument. [1]

(ii) Use a sign change argument to show that Joe's statement is correct. [2]

The graph of $y = x^3 - 3x^2 - 10x + 25$ in Fig. 11 shows that there is a root between 2 and 3.

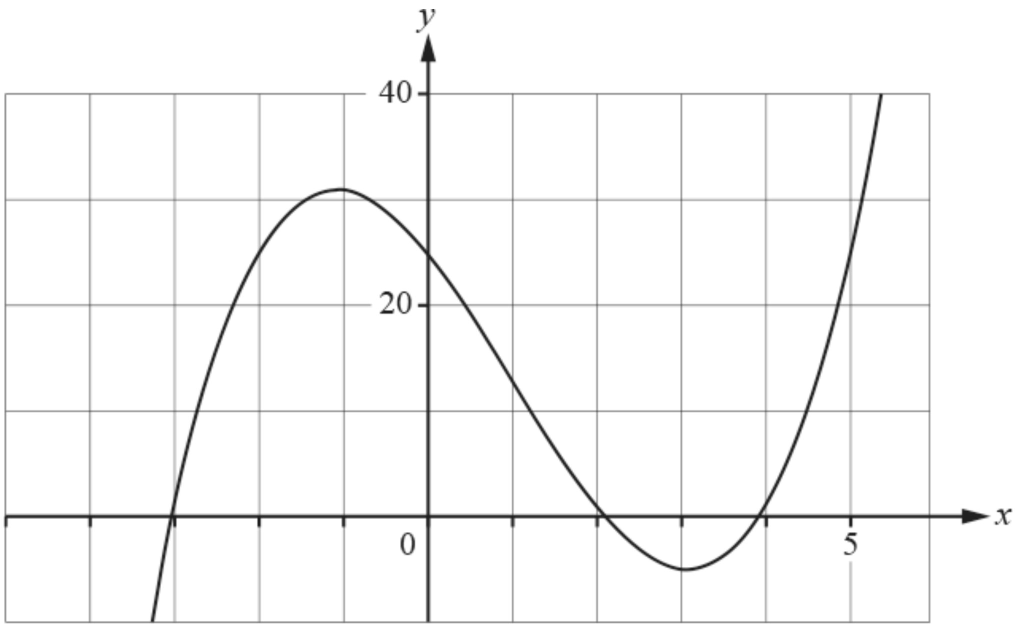


Fig.11

(c) Joe uses $x_0 = 3$ to attempt to find this root.

(i) Find x_1, x_2 and x_3. [2]

(ii) Explain why the Newton-Raphson method fails to give the required root in this case. [2]

(d) Explain what Joe should do to find the root of the equation between 2 and 3. [1]

4. By considering a change of sign, show that the equation $e^x - 5x^3 = 0$ has a root between 0 and 1. [2]

5. Rebecca is looking for the root of the equation

$$\sin 2x^2 - \cos 5x = 0$$

that lies between 0.2 and 0.3.
She uses the standard small angle approximations for $\sin\theta$ and $\cos\theta$ to find an estimate.

(a) Show that Rebecca's method gives an estimate of 0.26 when rounded to 2 decimal places. [3]

(b) Use a change of sign method to determine whether this value gives an estimate of the root which is correct to 2 decimal places.

[2]

Rebecca then uses the Newton-Raphson method to find another estimate for the root.

(c) Show that using $x_0 = 0.2$ gives the value $x_1 = 0.291\,989\,3$ correct to 7 decimal places. [5]

(d) Continue this method, showing the result of each iteration, to find the root correct to 3 significant figures.

[3]

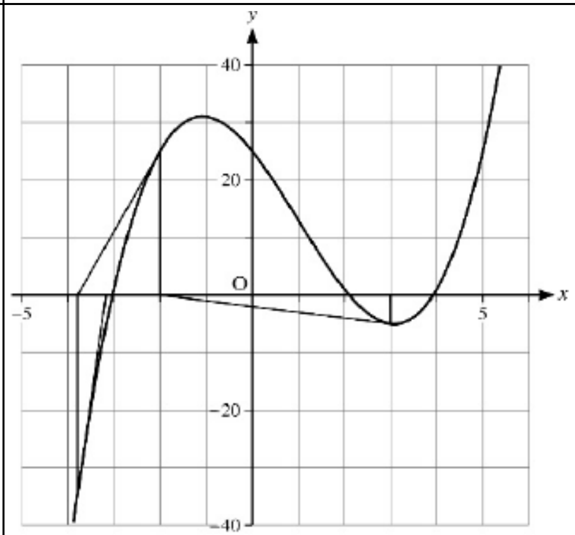
END OF QUESTION paper

Mark scheme

Question			Answer/Indicative content	Marks	Guidance
1		a	$f(-1) = (-1)^4 + (-1)^3 - 2(-1)^2 - 4(-1) - 2$ $= 1 - 1 - 2 + 4 - 2 = 0$	E1(AO1.1) [1]	
		b	f(1) = 1 + 1 − 2 − 4 − 2 = −6 or 'negative' f(2) = 16 + 8 − 8 − 8 − 2 = 6 or 'positive' change of sign ⇒ root between 1 and 2	B1(AO1.1) E1(AO2.4) [2]	both correct allow no mention of continuity of f AG
		c	long division or equating coeffts ⇒ $g(x) = x^3 - 2x - 2$ so $a = -2$, $b = -2$	M1(AO1.1) A1A1(AO2.2a 1.1) [3]	
		d	Clear explanation E.g. $f(x) = (x + 1)g(x)$ For the root of $f(x) = 0$ between 1 and 2, RHS is also zero hence $g(x) = 0$	E1(AO2.4) [1]	
		e	$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$ $= x_n - \frac{x_n^3 - 2x_n - 2}{3x_n^2 - 2}$	M1(AO1.1) E1(AO2.4)	

			$= \dfrac{3{x_n}^3 - 2x_n - {x_n}^3 + 2x_n + 2}{3{x_n}^2 - 2}$ $= \dfrac{2{x_n}^3 + 2}{3{x_n}^2 - 2}$ Root 1.769 (4sf)	A1(AO2.2a) [3]	AG BC
			Total	10	
2		a	f(0.1) = 3.897…and f(0.9) = −1.194… sign change so 0.1 < α < 0.9	B1(AO 1.1) E1(AO 1.2) [2]	
		b	$\left[\dfrac{dy}{dx} = \right] 2 - \dfrac{1}{x^2} + \dfrac{1}{x}$ $x_{r+1} = x_r - \dfrac{2x_r + \dfrac{1}{x_r} + \ln x_r - 4}{2 - \dfrac{1}{x_r^2} + \dfrac{1}{x_r}}$ $x_{r+1} = x_r - \dfrac{2x_r^3 + x_r + x_r^2(\ln x_r - 4)}{2x_r^2 - 1 + x_r}$	B1(AO 1.1) M1(AO 1.1) A1(AO 2.2a) [3]	AG
		c	(A) $x_1 = -0.52359…$	B1(AO 1.1) E1(AO 2.4)	

			which is negative so the iteration stops because f(x) is undefined for $x < 0$	[2]	
		c	(B) gradient is zero at 0.5 so tangent never touches x-axis	B1(AO 1.1) E1(AO 2.4) [2]	
		c	(C) 2.4497, 1.4667, 1.4675, 1.4675 *or* 0.6 is to the right of the turning point so converges to larger root	B1(AO 1.1) E1(AO 2.4) [2]	
			Total	11	
3		a	$f(x) = x^3 - 3x^2 - 10x + 25 \Rightarrow f'(x) = 3x^2 - 6x - 10$, so the N-R formula gives $x_{n+1} = x_n - \dfrac{x_n^3 - 3x_n^2 - 10x_n + 25}{3x_n^2 - 6x_n - 10}$	E1(AO2.1) [1]	**AG** Must be clear that denominator is derivative of numerator
		b	(i) Not valid: the sequence may decrease further, far enough to change the first 3 figures	B1(AOs 2.3)	Reason for 'not valid'

			(ii) f (3.915) = – 0.1255… and f(3.925) = 2.03… x 10^{-4} Change of sign shows that there is a root in the interval (3.915, 3.925) so the root is 3.92 to 2dp	[1] M1(AO2.1) A1(AO2.2a) [2]	needed Both calculations Complete argument needed	Allow use of any two values closer to 3.92 that give sign change
		c	(i) $x_0 = 3 \Rightarrow x_1 = -2$ $x_2 = -3.785\ 71.$ and $x_3 = -3.168\ 34.$	B1(AO1.1b) B1(AO1.1b) [2]	Correct first iteration x_2 and x_3 correct to at least 3dp	
			(ii)  The initial value is close to a stationary point, so the tangent meets the x-axis far from the required root, and the sequence converges to the wrong root	B1(AO2.3) B1(AO2.4) [2]	Explanations do not need to include a sketch; if a sketch is included, ignore any inaccuracies if correct explanation is given; sketch with no explanation scores 0 ‘close to stationary point’ oe seen	

					‘converges to wrong root’ oe seen
		d	Choose starting value near the root and not near a stationary point, eg take $x_0 = 2$	E1(AO2.1) [1]	
			Total	9	
4			When x = 0 $e^0 - 5 \times 0^3 = 1 > 0$ When x = 1 $e^1 - 5 \times 1^3 = e - 5 < 0$ So [as the function is continuous and there is a change of sign] there is a root between 0 and 1	M1 (AO 1.1a) E1 (AO 2.2a) [2]	Attempting to evaluate the function at both values Conclusion from correct values **Examiner's Comments** This question was generally well answered with only a few candidates making an arithmetical error that cost a mark.
			Total	2	
5		a	Using $\sin\theta \approx \theta$ with $\theta = 2x^2$ and $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ with $\theta = 5x$ gives $2x^2 - \left(1 - \frac{1}{2}(5x)^2\right) = 0$ $\frac{29}{2}x^2 = 1 \Rightarrow x = \sqrt{\frac{2}{29}} = 0.2626K$ So estimate is 0.26 to 2 decimal places	M1 (AO 2.1) M1 (AO 2.1) A1 (AO 2.1)	Allow slip in $(5x)^2$ Attempt to solve for x **AG** Must be rounded

				[3]	to 2 dp following either exact answer or answer to more than 2 dp seen	
		b	Denoting $\sin 2x^2 - \cos 5x$ by $f(x)$: $f(0.255) = -0.1618\ldots$ and $f(0.265) = -0.1033\ldots$ These are both negative so the root does not lie between 0.255 and 0.265, so the estimate is not correct to 2 decimal places	M1 (AO 2.1) A1 (AO 2.2a) [2]	Search for sign change using their 2dp value ±0.005 Complete argument from correct figs	oe, e.g. comparing values of $\sin 2x^2$ and $\cos 5x$ at end-points
		c	$f'(x) = 4x\cos 2x^2 + 5\sin 5x$ $x_1 = 0.2 - \frac{\sin(2\times0.2^2) - \cos(5\times0.2)}{4\times0.2\cos(2\times0.2^2) + 5\sin(5\times0.2)}$ $= 0.2 - \frac{-0.460\,387\,6119}{5.004\,796\,289} = 0.291\,989\,28\ldots$ so $x_1 = 0.291\,989\,3$ correct to 7 dp	M1 (AO 1.1a) M1 (AO 1.1a) A1 (AO 1.1) M1 (AO 2.1) A1 (AO 2.1) [5]	Attempt to find $f'(x)$ Use of chain rule for $\sin 2x^2$ Derivative fully correct Use of iterative formula to find x_1 **AG** Answer must	

					follow from clear working
		d	$x_2 = 0.282\ 338\ 3$ $x_3 = 0.282\ 285\ 3$ Root is 0.282 correct to 3sf	M1 (AO 1.1a) A1 (AO 1.1) B1 (AO 2.2b) [3]	Finds at least one more iteration For correct x_3
			Total	13	