Questions

Q1.

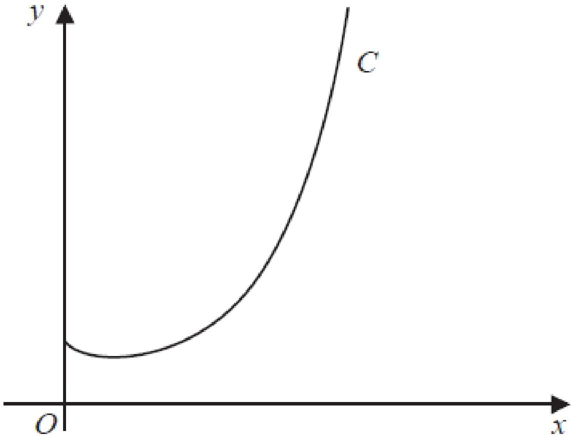


Figure 8

Figure 8 shows a sketch of the curve C with equation $y = x^x$, $x > 0$

(a) Find, by firstly taking logarithms, the x coordinate of the turning point of C.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

The point $P(\alpha, 2)$ lies on C.

(b) Show that $1.5 < \alpha < 1.6$

(2)

A possible iteration formula that could be used in an attempt to find α is

$$x_{n+1} = 2x_n^{1-x_n}$$

Using this formula with $x_1 = 1.5$

(c) find x_4 to 3 decimal places,

(2)

(d) describe the long-term behaviour of x_n

(2)

(Total for question = 11 marks)

Q2.

The sequence u_1, u_2, u_3,... is defined by

$$u_{n+1} = k - \frac{24}{u_n} \qquad u_1 = 2$$

where k is an integer.

Given that $u_1 + 2u_2 + u_3 = 0$

(a) show that

$$3k^2 - 58k + 240 = 0$$

(3)

(b) Find the value of k, giving a reason for your answer.

(2)

(c) Find the value of u_3

(1)

(Total for question = 6 marks)

Q3.

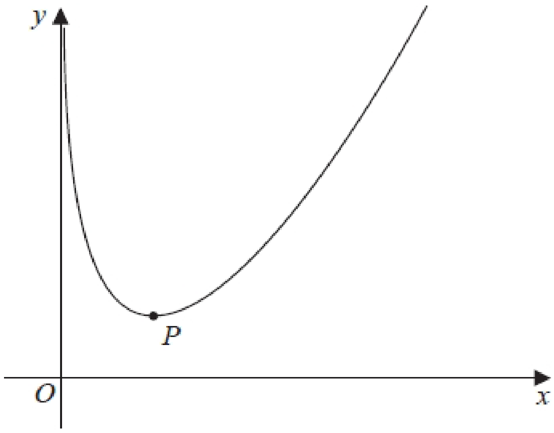


Figure 1

Figure 1 shows a sketch of the curve *C* with equation

$$y = \frac{4x^2 + x}{2\sqrt{x}} - 4\ln x \qquad x > 0$$

(a) Show that

$$\frac{dy}{dx} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$$

(4)

The point *P*, shown in Figure 1, is the minimum turning point on *C*.

(b) Show that the *x* coordinate of *P* is a solution of

$$x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{\frac{2}{3}}$$

(3)

(c) Use the iteration formula

$$x_{n+1} = \left(\frac{4}{3} - \frac{\sqrt{x_n}}{12}\right)^{\frac{2}{3}} \qquad \text{with } x_1 = 2$$

to find (i) the value of x_2 to 5 decimal places,
(ii) the *x* coordinate of *P* to 5 decimal places.

(3)

(Total for question = 10 marks)

Q4.

$$f(x) = \ln(2x - 5) + 2x^2 - 30, \quad x > 2.5$$

(a) Show that $f(x) = 0$ has a root α in the interval [3.5, 4]

(2)

A student takes 4 as the first approximation to α.

Given $f(4) = 3.099$ and $f'(4) = 16.67$ to 4 significant figures,

(b) apply the Newton-Raphson procedure once to obtain a second approximation for α, giving your answer to 3 significant figures.

(2)

(c) Show that α is the only root of $f(x) = 0$

(2)

(Total for question = 6 marks)

Q5.

The equation $2x^3 + x^2 - 1 = 0$ has exactly one real root.

(a) Show that, for this equation, the Newton-Raphson formula can be written

$$x_{n+1} = \frac{4x_n^3 + x_n^2 + 1}{6x_n^2 + 2x_n}$$

(3)

Using the formula given in part (a) with $x_1 = 1$

(b) find the values of x_2 and x_3

(2)

(c) Explain why, for this question, the Newton-Raphson method cannot be used with $x_1 = 0$

(1)

(Total for question = 6 marks)

Q6.

The curve with equation $y = 2\ \ln(8 - x)$ meets the line $y = x$ at a single point, $x = \alpha$.

(a) Show that $3 < \alpha < 4$

(2)

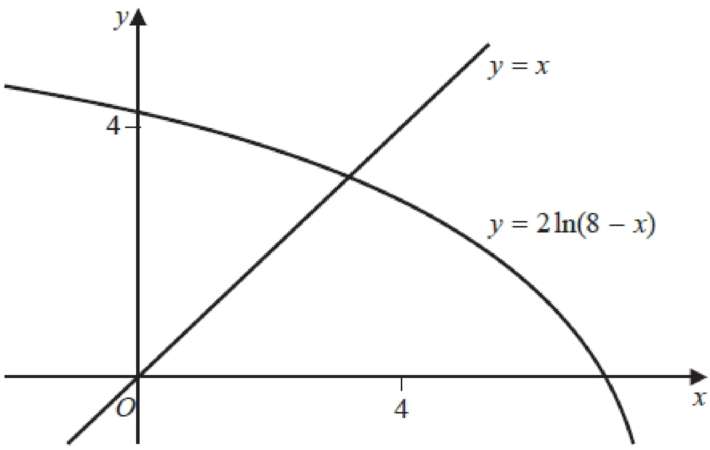


Figure 2

Figure 2 shows the graph of $y = 2\ \ln(8 - x)$ and the graph of $y = x$.

A student uses the iteration formula

$$x_{n+1} = 2\ln(8 - x_n), \quad n \in \mathbb{N}$$

in an attempt to find an approximation for α.

Using the graph and starting with $x_1 = 4$

(b) determine whether or not this iteration formula can be used to find an approximation for α, justifying your answer.

(2)

(Total for question = 4 marks)

Mark Scheme

Q1.

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$y = x^x \Rightarrow \ln y = x \ln x$	M1	This mark is for a method to find the x-coordinate of the turning point of C by taking logarithms
	$\ln y = x \ln x \Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln x + 1$	M1	This mark is given for a method using implicit differentiation
		A1	This mark is given for a correct expression for $\frac{1}{y}\frac{dy}{dx}$
	Setting $\frac{dy}{dx} = 0$, $\ln x + 1 = 0$	M1	This mark is given for a method for finding the turning point of C by setting $\frac{dy}{dx} = 0$
	$x = e^{-1}$	A1	This mark is given for correctly finding a value for the x-coordinate of the turning point of C
(b)	$1.5^{1.5} = 1.837\ldots$, $1.6^{1.6} = 2.121\ldots$	M1	This mark is given for substituting 1.5 and 1.6 into $y = x^x$
	The curve C contains the points (1.5, 1.8) and (1.6, 2.1). At P, $y = 2$ Since C is continuous, $1.5 < \alpha < 1.6$	A1	This mark is given for a valid explanation that C contains the points (1.5, 1.8) and (1.6, 2.1) and is continuous
(c)	$x_1 = 1.5$ $x_2 = 2 \times 1.5^{-0.5} = 1.633$	M1	This mark is given for finding a correct value for x_2
	$x_3 = 2 \times 1.633^{-0.633} = 1.466$ $x_4 = 2 \times 1.466^{-0.466} = 1.673$	A1	This mark is given for finding a correct value for x_4
(d)	For example: x_n oscillates is periodic is non-convergent	B1	This mark is given for a valid statement about the long-term behaviour of x_n
	between 1 and 2	B1	This mark is given for stating that the behaviour is between 1 and 2
			(Total 11 marks)

Q2.

Question	Scheme	Marks	AOs
(a)	$u_2 = k-12,\ u_3 = k - \frac{24}{k-12}$	M1	1.1b
	$u_1 + 2u_2 + u_3 = 0 \Rightarrow 2 + 2(k-12) + k - \frac{24}{k-12} = 0$	dM1	1.1b
	$\Rightarrow 3k - 22 - \frac{24}{k-12} = 0 \Rightarrow (3k-22)(k-12) - 24 = 0$ $\Rightarrow 3k^2 - 36k - 22k + 264 - 24 = 0$ $\Rightarrow 3k^2 - 58k + 240 = 0$ *	A1*	2.1
		(3)	
(b)	$k = 6, \left(\frac{40}{3}\right)$	M1	1.1b
	$k = 6$ as k must be an integer	A1	2.3
		(2)	
(c)	$(u_3 =)10$	B1	2.2a
		(1)	
		(6 marks)	
	Notes		

(a)

M1: Attempts to apply the sequence formula once for either u_2 or u_3.

Usually for $u_2 = k - \frac{24}{2}$ o.e. but could be awarded for $u_3 = k - \frac{24}{their\ "u_2"}$

dM1: Award for

- attempting to apply the sequence formula to find both u_2 **and** u_3
- using $2 + 2"u_2" + "u_3" = 0 \Rightarrow$ an equation in k. The u_3 may have been incorrectly adapted

A1*: Fully correct work leading to the printed answer.

There must be

- (at least) one correct intermediate line between $2 + 2(k-12) + k - \frac{24}{k-12} = 0$ (o.e.) and the given answer that shows how the fractions are "removed". E.g. $(3k-22)(k-12) - 24 = 0$
- no errors in the algebra. The = 0 may just appear at the answer line.

(b)

M1: Attempts to solve the quadratic which is implied by sight of $k = 6$.

This may be awarded for any of

- $3k^2 - 58k + 240 = (ak \pm c)(bk \pm d) = 0$ where $ab = 3, cd = 240$ followed by $k =$
- an attempt at the correct quadratic formula (or completing the square)
- a calculator solution giving at least $k = 6$

A1: Chooses $k = 6$ and gives a minimal reason

Examples of a minimal reason are

- 6 because it is an integer
- 6 because it is a whole number
- 6 because $\frac{40}{3}$ or $13.\dot{3}$ is not an integer

(c)

B1: Deduces the correct value of u_3.

Q3.

Question	Scheme	Marks	AOs
(a)	$\ln x \to \frac{1}{x}$	B1	1.1a
	Method to differentiate $\frac{4x^2+x}{2\sqrt{x}}$ – see notes	M1	1.1b
	E.g. $2\times\frac{3}{2}x^{\frac{1}{2}}+\frac{1}{2}\times\frac{1}{2}x^{-\frac{1}{2}}$	A1	1.1b
	$\frac{dy}{dx}=3\sqrt{x}+\frac{1}{4\sqrt{x}}-\frac{4}{x}=\frac{12x^2+x-16\sqrt{x}}{4x\sqrt{x}}$ *	A1*	2.1
		(4)	
(b)	$12x^2+x-16\sqrt{x}=0 \Rightarrow 12x^{\frac{3}{2}}+x^{\frac{1}{2}}-16=0$	M1	1.1b
	E.g. $12x^{\frac{3}{2}}=16-\sqrt{x}$	dM1	1.1b
	$x^{\frac{3}{2}}=\frac{4}{3}-\frac{\sqrt{x}}{12} \Rightarrow x=\left(\frac{4}{3}-\frac{\sqrt{x}}{12}\right)^{\frac{2}{3}}$ *	A1*	2.1
		(3)	
(c)	$x_2=\sqrt[3]{\left(\frac{4}{3}-\frac{\sqrt{2}}{12}\right)^2}$	M1	1.1b
	x_2 = awrt 1.13894	A1	1.1b
	x = 1.15650	A1	2.2a
		(3)	
			(10 marks)

Notes:

(a)

B1: Differentiates $\ln x \to \frac{1}{x}$ seen or implied

M1: Correct method to differentiate $\frac{4x^2+x}{2\sqrt{x}}$:

Look for $\frac{4x^2+x}{2\sqrt{x}} \to ...x^{\frac{3}{2}}+...x^{\frac{1}{2}}$ being then differentiated to $Px^{\frac{1}{2}}+...$ or $...+Qx^{-\frac{1}{2}}$

Alternatively uses the quotient rule on $\frac{4x^2+x}{2\sqrt{x}}$.

Condone slips but if rule is not quoted expect $\left(\frac{dy}{dx}\right)=\frac{2\sqrt{x}(Ax+B)-(4x^2+x)Cx^{-\frac{1}{2}}}{(2\sqrt{x})^2}(A,B,C>0)$

But a correct rule may be implied by their u, v, u', v' followed by applying $\frac{vu'-uv'}{v^2}$ etc.

Alternatively uses the product rule on $(4x^2+x)(2\sqrt{x})^{-1}$

Condone slips but expect $\left(\frac{dy}{dx}\right)=Ax^{-\frac{1}{2}}(Bx+C)+D(4x^2+x)x^{-\frac{3}{2}}(A,B,C>0)$

In general condone missing brackets for the M mark. If they quote $u=4x^2+x$ and $v=2\sqrt{x}$ and don't make the differentiation easier, they can be awarded this mark for applying the correct rule. Also allow this mark if they quote the correct quotient rule but only have v rather than v^2 in the denominator.

A1: Correct differentiation of $\frac{4x^2+x}{2\sqrt{x}}$ although may not be simplified.

Examples: $\left(\frac{dy}{dx}\right)=\frac{2\sqrt{x}(8x+1)-(4x^2+x)x^{-\frac{1}{2}}}{(2\sqrt{x})^2}$, $\frac{1}{2}x^{-\frac{1}{2}}(8x+1)-\frac{1}{4}(4x^2+x)x^{-\frac{3}{2}}$, $2\times\frac{3}{2}x^{\frac{1}{2}}+\frac{1}{2}\times\frac{1}{2}x^{-\frac{1}{2}}$

A1*: Obtains $\frac{dy}{dx}=\frac{12x^2+x-16\sqrt{x}}{4x\sqrt{x}}$ via $3\sqrt{x}+\frac{1}{4\sqrt{x}}-\frac{4}{x}$ or a correct application of the quotient or product rule and with sufficient working shown to reach the printed answer.
There must be no errors e.g. missing brackets.

(b)
M1: Sets $12x^2+x-16\sqrt{x}=0$ and divides by $\sqrt{x}$ or equivalent e.g. divides by x and multiplies by $\sqrt{x}$

dM1: Makes the term in $x^{\frac{3}{2}}$ the subject of the formula
A1*: A correct and rigorous argument leading to the given solution.

Alternative - working backwards:

$$x=\left(\frac{4}{3}-\frac{\sqrt{x}}{12}\right)^{\frac{2}{3}}\Rightarrow x^{\frac{3}{2}}=\frac{4}{3}-\frac{\sqrt{x}}{12}\Rightarrow 12x^{\frac{3}{2}}=16-\sqrt{x}\Rightarrow 12x^2=16\sqrt{x}-x\Rightarrow 12x^2-16\sqrt{x}+x=0$$

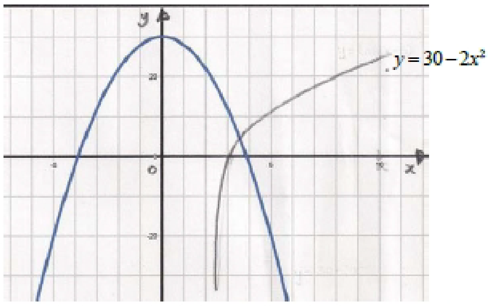
M1: For raising to power of 3/2 both sides. dM1: Multiplies through by $\sqrt{x}$. A1: Achieves printed answer and makes a minimal comment e.g. tick, #, QED, true etc.

(c)

M1: Attempts to use the iterative formula with $x_1=2$. This is implied by sight of $x_2=\left(\frac{4}{3}-\frac{\sqrt{2}}{12}\right)^{\frac{2}{3}}$ or awrt 1.14

A1: x_2 = awrt 1.13894
A1: Deduces that x = 1.15650

Q4.

Question	Scheme	Marks	AOs
(a)	$f(3.5) = -4.8,\ f(4) = (+)3.1$	M1	1.1b
	Change of sign and function continuous in interval $[3.5, 4] \Rightarrow$ Root *	A1*	2.4
		(2)	
(b)	Attempts $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} \Rightarrow \quad x_1 = 4 - \dfrac{3.099}{16.67}$	M1	1.1b
	$x_1 = 3.81$	A1	1.1b
	$y = \ln(2x-5)$	(2)	
(c)	 Attempts to sketch both $y = \ln(2x-5)$ and $y = 30 - 2x^2$	M1	3.1a
	States that $y = \ln(2x-5)$ meets $y = 30 - 2x^2$ in just one place, therefore $y = \ln(2x-5) = 30 - 2x$ has just one root $\Rightarrow f(x) = 0$ has just one root	A1	2.4
		(2)	
			(6 marks)

Notes:

(a)

M1: Attempts f(x) at both $x = 3.5$ and $x = 4$ with at least one correct to 1 significant figure

A1*: f(3.5) and f(4) correct to 1 sig figure (rounded or truncated) with a correct reason and conclusion. A reason could be change of sign, or $f(3.5)\times f(4) < 0$ or similar with f(x) being continuous in this interval. A conclusion could be 'Hence root' or 'Therefore root in interval'

(b)

M1: Attempts $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ evidenced by $x_1 = 4 - \dfrac{3.099}{16.67}$

A1: Correct answer only $x_1 = 3.81$

(c)

M1: For a valid attempt at showing that there is only one root. This can be achieved by

- Sketching graphs of $y = \ln(2x-5)$ and $y = 30 - 2x^2$ on the same axes
- Showing that $f(x) = \ln(2x-5) + 2x^2 - 30$ has no turning points
- Sketching a graph of $f(x) = \ln(2x-5) + 2x^2 - 30$

A1: Scored for correct conclusion

Q5.

Question	Scheme	Marks	AOs
	The equation $2x^3+x^2-1=0$ has exactly one real root		
(a)	$\{f(x)=2x^3+x^2-1 \Rightarrow\}$ $f'(x)=6x^2+2x$	B1	1.1b
	$\left\{x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)} \Rightarrow\right\}$ $\{x_{n+1}\}=x_n-\dfrac{2x_n^3+x_n^2-1}{6x_n^2+2x_n}$	M1	1.1b
	$=\dfrac{x_n(6x_n^2+2x_n)-(2x_n^3+x_n^2-1)}{6x_n^2+2x_n} \Rightarrow x_{n+1}=\dfrac{4x_n^3+x_n^2+1}{6x_n^2+2x_n}$ *	A1*	2.1
		(3)	
(b)	$\{x_1=1 \Rightarrow\}$ $x_2=\dfrac{4(1)^3+(1)^2+1}{6(1)^2+2(1)}$ or $x_2=1-\dfrac{2(1)^3+(1)^2-1}{6(1)^2+2(1)}$	M1	1.1b
	$\Rightarrow x_2=\dfrac{3}{4},\ x_3=\dfrac{2}{3}$	A1	1.1b
		(2)	
(c)	Accept any reasons why the Newton-Raphson **method** cannot be used with $x_1=0$ which refer or *allude* to either the stationary point or the tangent. E.g. • There is a stationary point at $x=0$ • Tangent to the curve (or $y=2x^3+x^2-1$) would not meet the x-axis • Tangent to the curve (or $y=2x^3+x^2-1$) is horizontal	B1	2.3
		(1)	
		(6 marks)	

Notes for Question	
(a)	
B1:	States that $f'(x)=6x^2+2x$ or states that $f'(x_n)=6x_n^{\ 2}+2x_n$ (Condone $\frac{dy}{dx}=6x^2+2x$)
M1:	Substitutes $f(x_n)=2x_n^{\ 3}+x_n^{\ 2}-1$ and their $f'(x_n)$ into $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$
A1*:	A correct intermediate step of making a common denominator which leads to the given answer
Note:	Allow B1 if $f'(x)=6x^2+2x$ is applied as $f'(x_n)$ (or $f'(x)$) in the NR formula $\{x_{n+1}\}=x_n-\frac{f(x_n)}{f'(x_n)}$
Note:	Allow M1A1 for • $x_{n+1}=x-\frac{2x^3+x^2-1}{6x^2+2x}=\frac{x(6x^2+2x)-(2x^3+x^2-1)}{6x^2+2x} \Rightarrow x_{n+1}=\frac{4x_n^{\ 3}+x_n^{\ 2}+1}{6x_n^{\ 2}+2x_n}$
Note	Condone $x=x-\frac{2x^3+x^2-1}{"6x^2+2x"}$ for M1
Note	Condone $x_n-\frac{2x_n^{\ 3}+x_n^{\ 2}-1}{"6x_n^{\ 2}+2x_n"}$ or $x-\frac{2x^3+x^2-1}{"6x^2+2x"}$ (i.e. no $x_{n+1}=...$) for M1
Note:	Give M0 for $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$ followed by $x_{n+1}=2x_n^{\ 3}+x_n^{\ 2}-1-\frac{2x_n^{\ 3}+x_n^{\ 2}-1}{6x_n^{\ 2}+2x_n}$
Note:	Correct notation, i.e. x_{n+1} and x_n must be seen in their final answer for A1*
(b)	
M1:	An attempt to use the given or their formula once. Can be implied by $\frac{4(1)^3+(1)^2+1}{6(1)^2+2(1)}$ or 0.75 o.e.
Note:	Allow one slip in substituting $x_1=1$
A1:	$x_2=\frac{3}{4}$ and $x_3=\frac{2}{3}$
Note:	Condone $x_2=\frac{3}{4}$ and $x_3=$ awrt 0.667 for A1
Note:	Condone $\frac{3}{4},\frac{2}{3}$ listed in a correct order ignoring subscripts
(c)	
B1:	See scheme
Note:	Give B0 for the following isolated reasons: e.g. • You cannot divide by 0 • The fraction (or the NR formula) is undefined at $x=0$ • At $x=0$, $f'(x_1)=0$ • x_1 cannot be 0 • $6x^2+2x$ cannot be 0 • the denominator is 0 which cannot happen • if $x_1=0$, $6x^2+2x=0$

Q6.

Question	Scheme	Marks	AOs
(a)	Attempts $\text{f}(3)=$ and $\text{f}(4)=$ where $\text{f}(x)=\pm\left(2\ln(8-x)-x\right)$	M1	2.1
	$\text{f}(3)=\left(2\ln(5)-x\right)=(+)0.22$ and $\text{f}(4)=\left(2\ln(4)-4\right)=-1.23$ Change of sign and function continuous in interval $[3,4]\Rightarrow$ Root *	A1*	2.4
		(2)	
(b)	For annotating the graph by drawing a cobweb diagram starting at $x_1=4$ It should have at least two spirals	M1	2.4
	Deduces that the iteration formula **can be used** to find an approximation for α because **the cobweb spirals inwards** for the cobweb diagram	A1	2.2a
		(2)	
			(4 marks)

Notes:

(a)

M1: Attempts $\text{f}(3)=$ and $\text{f}(4)=$ where $\text{f}(x)=\pm\left(2\ln(8-x)-x\right)$ or alternatively **compares** $2\ln 5$ to 3 and $2\ln 4$ to 4. This is not routine and cannot be scored by substituting 3 and 4 in both functions

A1: Both values (calculations) correct to at least 1 sf with correct explanation and conclusion. (See underlined statements)
When comparing terms, allow reasons to be $2ln8=3.21>3,\ 2\ln4=2.77<4$ or similar

(b)

M1: For an attempt at using a cobweb diagram. Look for 5 or more correct straight lines. It may not start at 4 but it must show an understanding of the method. **If there is no graph then it is M0 A0**
A1: For a correct attempt starting at 4 and deducing that the iteration **can be used** as the iterations **converge to the root**. You must statement that it can be used with a suitable reason. Suitable reasons could be " it spirals inwards", it gets closer to the root", it converges "