INTEGRATION Answers

1 **a** when $y = 3$, $x = \frac{1}{3}$

area $0 \le x \le \frac{1}{3} = 3 \times \frac{1}{3} = 1$

area $\frac{1}{3} < x \le 3 = \int_{\frac{1}{3}}^{3} \frac{1}{x}\,dx$

$= [\ln|x|]_{\frac{1}{3}}^{3}$

$= \ln 3 - \ln \frac{1}{3} = \ln \frac{3}{\frac{1}{3}} = \ln 9$

total area $= 1 + \ln 9$

b volume $0 \le x \le \frac{1}{3}$ = volume of cylinder

$= \pi \times 3^2 \times \frac{1}{3} = 3\pi$

volume $\frac{1}{3} < x \le 3 = \pi\int_{\frac{1}{3}}^{3} (\frac{1}{x})^2\,dx$

$= \pi\int_{\frac{1}{3}}^{3} x^{-2}\,dx = \pi[-x^{-1}]_{\frac{1}{3}}^{3}$

$= \pi[-\frac{1}{3} - (-3)] = \frac{8}{3}\pi$

total volume $= 3\pi + \frac{8}{3}\pi = \frac{17}{3}\pi$

2 **a**

x	0	0.5	1	1.5	2	2.5	3	3.5	4
$x \sec(\frac{1}{3}x)$	0	0.507	1.058	1.709	2.545	3.718	5.552	8.901	17.004

i $\approx \frac{1}{2} \times 2 \times [0 + 17.004 + 2(2.545)] = 22.1$ (3sf)

ii $\approx \frac{1}{2} \times 1 \times [0 + 17.004 + 2(1.058 + 2.545 + 5.552)] = 17.7$ (3sf)

iii $\approx \frac{1}{2}\times0.5\times[0 + 17.004 + 2(0.507 + 1.058 + 1.709 + 2.545 + 3.718 + 5.552 + 8.901)] = 16.2$ (3sf)

b e.g. halving the interval width leads to the estimate reducing by 4.4, then 1.5
further halving might lead to reductions of approximately 0.5 and 0.2 so $I \approx 15.5$

3 **a** $-\frac{d\theta}{dt} = k\theta$

$\int \frac{1}{\theta}\,d\theta = \int -k\,dt$

$\ln|\theta| = -kt + c$

$\theta = e^{-kt+c} = e^c \times e^{-kt}$

$\therefore\ \theta = Ae^{-kt}$

$t = 0,\ \theta = 25 - 10 = 15$

$\therefore\ A = 15$

$\therefore\ \theta = 15e^{-kt}$

b $t = 30,\ \theta = 20 - 10 = 10$

$\therefore\ 10 = 15e^{-30k}$

$k = -\frac{1}{30}\ln\frac{2}{3} = 0.0135$ (3sf)

c $\theta = 15 - 10 = 5$

$\therefore\ 5 = 15e^{-0.01352t}$

$t = -\frac{1}{0.01352}\ln\frac{1}{3}$

$= 81.3$ minutes (3sf)

4 **a** $\sin^4 x \equiv (\sin^2 x)^2$

$\equiv [\frac{1}{2}(1 - \cos 2x)]^2$

$\equiv \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$

$\equiv \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}[\frac{1}{2}(1 + \cos 4x)]$

$\equiv \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$

$\therefore\ p = \frac{3}{8},\ q = -\frac{1}{2},\ r = \frac{1}{8}$

b $= \int_0^{\frac{\pi}{2}} (\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x)\,dx$

$= [\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x]_0^{\frac{\pi}{2}}$

$= (\frac{3}{16}\pi - 0 + 0) - (0 - 0 + 0)$

$= \frac{3}{16}\pi$

5 **a** $\int y^{-3}\ dy = \int x\ dx$

$-\frac{1}{2}y^{-2} = \frac{1}{2}x^2 + c$

$y^{-2} = k - x^2$

$y^2 = \frac{1}{k - x^2}$

b $y = \frac{1}{2}$ when $x = 1$

$\therefore\ \frac{1}{4} = \frac{1}{k-1}$

$k = 5$

$\therefore\ y^2 = \frac{1}{5 - x^2}$

6 **a** $x = e^u\ \therefore\ u = \ln x,\ \frac{dx}{du} = e^u$

$\int \frac{2+\ln x}{x^2}\ dx = \int \frac{2+u}{(e^u)^2} \times e^u\ du$

$= \int (2 + u)e^{-u}\ du$

b $x = e^u$

$x = 1 \Rightarrow u = 0$

$x = e \Rightarrow u = 1$

$\therefore\ \int_1^e \frac{2+\ln x}{x^2}\ dx = \int_0^1 (2 + u)e^{-u}\ du$

$v = 2 + u,\ \frac{dv}{du} = 1;\ \frac{dw}{du} = e^{-u},\ w = -e^{-u}$

$= [-(2 + u)e^{-u}]_0^1 + \int_0^1 e^{-u}\ du$

$= [-(2 + u)e^{-u} - e^{-u}]_0^1$

$= (-3e^{-1} - e^{-1}) - (-2 - 1)$

$= 3 - 4e^{-1}$

7 **a** $x = 0 \Rightarrow \cos 2t = 0 \Rightarrow t = \frac{\pi}{4}$

$y = 0 \Rightarrow \tan t = 0 \Rightarrow t = 0$

$x = \cos 2t\ \therefore\ \frac{dx}{dt} = -2\sin 2t$

shaded area $= \int_{\frac{\pi}{4}}^{0} \tan t \times (-2\sin 2t)\ dt$

$= \int_0^{\frac{\pi}{4}} \frac{\sin t}{\cos t} \times 4\sin t\cos t\ dt$

$= \int_0^{\frac{\pi}{4}} 4\sin^2 t\ dt$

b $= \int_0^{\frac{\pi}{4}} (2 - 2\cos 2t)\ dt$

$= [2t - \sin 2t]_0^{\frac{\pi}{4}}$

$= (\frac{\pi}{2} - 1) - (0 - 0)$

$= \frac{\pi}{2} - 1$

c **i** $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

ii $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$y^2 = \tan^2 t = \frac{\sin^2 t}{\cos^2 t} = \frac{\frac{1}{2}(1-\cos 2t)}{\frac{1}{2}(1+\cos 2t)}$

$\therefore\ y^2 = \frac{1-x}{1+x}$

8 **a** $\frac{6-2x^2}{(x+1)^2(x+3)} \equiv \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x+3}$

$6 - 2x^2 \equiv A(x + 3) + B(x + 1)(x + 3) + C(x + 1)^2$

$x = -1 \Rightarrow 4 = 2A \Rightarrow A = 2$

$x = -3 \Rightarrow -12 = 4C \Rightarrow C = -3$

coeffs $x^2 \Rightarrow -2 = B + C \Rightarrow B = 1$

b $x = 0 \Rightarrow y = 2$

$f(x) = 2(x + 1)^{-2} + (x + 1)^{-1} - 3(x + 3)^{-1}$

$f'(x) = -4(x + 1)^{-3} - (x + 1)^{-2} + 3(x + 3)^{-2}$

$\therefore\ f'(0) = -4 - 1 + \frac{1}{3} = -\frac{14}{3}$

$\therefore\ y - 2 = -\frac{14}{3}(x - 0)$

$3y - 6 = -14x$

$14x + 3y = 6$

c $= \int_0^1 \left(\frac{2}{(x+1)^2} + \frac{1}{x+1} - \frac{3}{x+3}\right) dx$

$= [-2(x + 1)^{-1} + \ln|x + 1| - 3\ln|x + 3|]_0^1$

$= (-1 + \ln 2 - 3\ln 4) - (-2 + 0 - 3\ln 3)$

$= 1 - 5\ln 2 + 3\ln 3$