INTEGRATION *Answers*

1 **a** $\dfrac{1}{x^2-3x+2} \equiv \dfrac{A}{x-1} + \dfrac{B}{x-2}$

$1 \equiv A(x-2) + B(x-1)$

$x = 1 \;\Rightarrow\; 1 = -A \;\Rightarrow\; A = -1$

$x = 2 \qquad\qquad\; \Rightarrow\; B = 1$

$\therefore\; \dfrac{1}{x^2-3x+2} \equiv \dfrac{1}{x-2} - \dfrac{1}{x-1}$

b $\displaystyle\int_3^4 \frac{1}{x^2-3x+2}\,dx = \int_3^4 \left(\frac{1}{x-2} - \frac{1}{x-1}\right) dx$

$= [\ln|x-2| - \ln|x-1|\,]_3^4$

$= (\ln 2 - \ln 3) - (0 - \ln 2)$

$= 2\ln 2 - \ln 3 = \ln \frac{2^2}{3}$

$= \ln \frac{4}{3}$ $\quad [\, a = 4,\; b = 3\,]$

2 $\displaystyle= \int_0^{\frac{\pi}{6}} \tfrac{1}{2}[\cos 4x + \cos(-2x)]\;dx$

$\displaystyle= \int_0^{\frac{\pi}{6}} (\tfrac{1}{2}\cos 4x + \tfrac{1}{2}\cos 2x)\;dx$

$= [\tfrac{1}{8}\sin 4x + \tfrac{1}{4}\sin 2x]_0^{\frac{\pi}{6}}$

$= (\tfrac{1}{8}\times\tfrac{\sqrt{3}}{2} + \tfrac{1}{4}\times\tfrac{\sqrt{3}}{2}) - (0)$

$= \tfrac{3}{16}\sqrt{3}$

3 **a**

$$\begin{array}{r|l} & x + 2 \\ \hline x-1 & x^2 + x - 1 \\ & x^2 - x \\ \hline & \quad 2x - 1 \\ & \quad 2x - 2 \\ \hline & \qquad\quad 1 \end{array}$$

quotient: $x + 2$, remainder: 1

b $\displaystyle\int \frac{x^2+x-1}{x-1}\,dx = \int \left(x + 2 + \frac{1}{x-1}\right) dx$

$= \tfrac{1}{2}x^2 + 2x + \ln|x-1| + c$

4 **a** $\displaystyle= \pi\int_1^4 \left(2 - \frac{1}{\sqrt{x}}\right)^2 dx$

$\displaystyle= \pi\int_1^4 (4 - 4x^{-\frac{1}{2}} + x^{-1})\;dx$

$= \pi[4x - 8x^{\frac{1}{2}} + \ln|x|\,]_1^4$

$= \pi[(16 - 16 + \ln 4) - (4 - 8 + 0)]$

$= \pi(4 + \ln 4) = \pi(4 + \ln 2^2)$

$= \pi(4 + 2\ln 2) = 2\pi(2 + \ln 2)$

b $= 10^3 \times 2\pi(2 + \ln 2)$

$= 2000\pi(2 + \ln 2) = 16\,900\text{ cm}^3$

5 **a** $u = \ln x,\; \dfrac{du}{dx} = \dfrac{1}{x};\;\; \dfrac{dv}{dx} = x,\; v = \tfrac{1}{2}x^2$

$\displaystyle\int x\ln x\;dx = \tfrac{1}{2}x^2\ln x - \int \tfrac{1}{2}x\;dx$

$= \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 + c$

$= \tfrac{1}{4}x^2(2\ln x - 1) + c$

b $\displaystyle\int \frac{1}{y}\;dy = \int x\ln x\;dx$

$\ln|y| = \tfrac{1}{4}x^2(2\ln x - 1) + c$

$y > 0 \;\therefore\; \ln y = \tfrac{1}{4}x^2(2\ln x - 1) + c$

$y = 4$ when $x = 2$

$\therefore\; \ln 4 = 2\ln 2 - 1 + c, \qquad c = 1$

$\therefore\; \ln y = \tfrac{1}{4}x^2(2\ln x - 1) + 1$

when $x = 1$, $\ln y = \tfrac{1}{4}(0 - 1) + 1 = \tfrac{3}{4}$

$\therefore\; y = e^{\frac{3}{4}}$

6 **a** $\displaystyle= \int_0^{\frac{\pi}{3}} \sec x\tan x\;dx$

$= [\sec x]_0^{\frac{\pi}{3}}$

$= 2 - 1$

$= 1$

b $u = \cos\theta \;\therefore\; \dfrac{du}{d\theta} = -\sin\theta$

$\theta = 0 \;\Rightarrow\; u = 1$

$\theta = \frac{\pi}{4} \;\Rightarrow\; u = \frac{1}{\sqrt{2}}$

$\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^4\theta}\;d\theta = \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u^4}\times(-1)\;du$

$\displaystyle= \int_{\frac{1}{\sqrt{2}}}^1 u^{-4}\;du$

$= [-\tfrac{1}{3}u^{-3}]_{\frac{1}{\sqrt{2}}}^1$

$= -\tfrac{1}{3}(1 - 2\sqrt{2})$

$= -\tfrac{1}{3} + \tfrac{2}{3}\sqrt{2}$ $\quad [\, a = -\tfrac{1}{3},\; b = \tfrac{2}{3}\,]$

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7 **a** $x = 0 \Rightarrow t = -\frac{1}{2}$

$x = 3 \Rightarrow t = 1$

b $x = 2t + 1 \ \therefore \ \frac{dx}{dt} = 2$

$\therefore$ area $= \int_{-\frac{1}{2}}^{1} \frac{1}{2-t} \times 2 \ dt$

$= \int_{-\frac{1}{2}}^{1} \frac{2}{2-t} \ dt$

$= [-2 \ln | 2 - t |]_{-\frac{1}{2}}^{1}$

$= -2(0 - \ln \frac{5}{2})$

$= 2 \ln \frac{5}{2}$

c $= \pi \int_{-\frac{1}{2}}^{1} (\frac{1}{2-t})^2 \times 2 \ dt$

$= 2\pi \int_{-\frac{1}{2}}^{1} (2 - t)^{-2} \ dt$

$= 2\pi[(2 - t)^{-1}]_{-\frac{1}{2}}^{1}$

$= 2\pi(1 - \frac{2}{5}) = \frac{6}{5}\pi$

8 **a** $u = 6x, \ \frac{du}{dx} = 6; \ \frac{dv}{dx} = \cos 3x, \ v = \frac{1}{3}\sin 3x$

$\int 6x \cos 3x \ dx = 2x \sin 3x - \int 2 \sin 3x \ dx$

$= 2x \sin 3x + \frac{2}{3}\cos 3x + c$

b $x = 2 \sin u \ \therefore \ \frac{dx}{du} = 2 \cos u$

$x = 0 \Rightarrow u = 0$

$x = \sqrt{3} \Rightarrow u = \frac{\pi}{3}$

$\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}} \ dx = \int_0^{\frac{\pi}{3}} \frac{1}{2\cos u} \times 2 \cos u \ du$

$= \int_0^{\frac{\pi}{3}} du$

$= [u]_0^{\frac{\pi}{3}}$

$= \frac{\pi}{3} - 0$

$= \frac{\pi}{3}$

9 **a** $A = \pi r^2 \ \therefore \ r = \sqrt{\frac{A}{\pi}}$

$P = 2\pi r = 2\pi\sqrt{\frac{A}{\pi}} = 2\sqrt{\pi A}$

$\frac{dA}{dt} = cP = c \times 2\sqrt{\pi A} = 2c\sqrt{\pi} \times \sqrt{A}$

$\therefore \ \frac{dA}{dt} = k\sqrt{A}$

b $\int A^{-\frac{1}{2}} \ dA = \int k \ dt$

$2A^{\frac{1}{2}} = kt + C$

$\sqrt{A} = \frac{1}{2}kt + \frac{1}{2}C = pt + q$

$\therefore \ A = (pt + q)^2$

c $t = 0, A = 25 \quad \therefore \ \sqrt{25} = 0 + q$

$q = 5$

$t = 20, A = 40 \quad \therefore \ \sqrt{40} = 20p + 5$

$p = \frac{2\sqrt{10} - 5}{20}$

$A = 50 \quad \Rightarrow \quad 50 = (\frac{2\sqrt{10} - 5}{20}t + 5)^2$

$t = (5\sqrt{2} - 5) \div \frac{2\sqrt{10} - 5}{20}$

$= 31.3$ minutes (3sf)

10 **a** $\frac{5x+1}{(1-x)(1+2x)} \equiv \frac{A}{1-x} + \frac{B}{1+2x}$

$5x + 1 \equiv A(1 + 2x) + B(1 - x)$

$x = 1 \quad \Rightarrow \ 6 = 3A \quad \Rightarrow \ A = 2$

$x = -\frac{1}{2} \quad \Rightarrow \ -\frac{3}{2} = \frac{3}{2}B \quad \Rightarrow \ B = -1$

$f(x) \equiv \frac{2}{1-x} - \frac{1}{1+2x}$

b $= \int_0^{\frac{1}{2}} (\frac{2}{1-x} - \frac{1}{1+2x}) \ dx$

$= [-2 \ln | 1 - x | - \frac{1}{2}\ln | 1 + 2x |]_0^{\frac{1}{2}}$

$= (-2 \ln \frac{1}{2} - \frac{1}{2}\ln 2) - (0 - 0)$

$= 2 \ln 2 - \frac{1}{2}\ln 2 = \frac{3}{2}\ln 2$

c $(1 - x)^{-1} =$

$1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(-x)^3 + \ldots$

$= 1 + x + x^2 + x^3 + \ldots$

$(1 + 2x)^{-1} =$

$1 + (-1)(2x) + \frac{(-1)(-2)}{2}(2x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(2x)^3 + \ldots$

$= 1 - 2x + 4x^2 - 8x^3 + \ldots$

$f(x) =$

$2(1 + x + x^2 + x^3) - (1 - 2x + 4x^2 - 8x^3) + \ldots$

$f(x) = 1 + 4x - 2x^2 + 10x^3 + \ldots$

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