

INTEGRATION

Answers

$$1 \quad \mathbf{a} \quad \frac{(x+4)}{(1+x)(2-x)} \equiv \frac{A}{1+x} + \frac{B}{2-x}$$

$$x+4 \equiv A(2-x) + B(1+x)$$

$$x=-1 \Rightarrow A=1, \quad x=2 \Rightarrow B=2$$

$$\therefore \frac{(x+4)}{(1+x)(2-x)} \equiv \frac{1}{1+x} + \frac{2}{2-x}$$

$$\mathbf{b} \quad \int \frac{1}{y} dy = \int \left(\frac{1}{1+x} + \frac{2}{2-x} \right) dx$$

$$\ln|y| = \ln|1+x| - 2\ln|2-x| + c$$

$$y=2 \text{ when } x=3$$

$$\therefore \ln 2 = \ln 4 - 0 + c$$

$$c = -\ln 2$$

$$\therefore \ln|y| = \ln|1+x| - 2\ln|2-x| - \ln 2$$

$$\left[y = \frac{1+x}{2(2-x)^2} \right]$$

$$3 \quad \frac{dy}{dx} = \frac{k}{x}$$

$$\frac{dy}{dx} = \frac{5}{3} \text{ when } x=3$$

$$\therefore \frac{5}{3} = \frac{k}{3}, \quad k=5$$

$$y = \int \frac{5}{x} dx$$

$$y = 5 \ln|x| + c$$

$$y=4 \text{ when } x=3$$

$$\therefore 4 = 5 \ln 3 + c$$

$$c = 4 - 5 \ln 3$$

$$\therefore y = 5 \ln|x| + 4 - 5 \ln 3$$

$$y = 5 \ln \left| \frac{x}{3} \right| + 4$$

$$2 \quad \frac{dy}{dx} = e^y \times e^x \cos x$$

$$\int e^{-y} dy = \int e^x \cos x dx$$

$$u = e^x, \quad \frac{du}{dx} = e^x; \quad \frac{dv}{dx} = \cos x, \quad v = \sin x$$

$$I = \int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$$

$$u = e^x, \quad \frac{du}{dx} = e^x; \quad \frac{dv}{dx} = \sin x, \quad v = -\cos x$$

$$I = e^x \sin x - (-e^x \cos x + \int e^x \cos x dx)$$

$$2I = e^x \sin x + e^x \cos x + C$$

$$-e^{-y} = \frac{1}{2} e^x (\sin x + \cos x) + c$$

$$e^{-y} = k - \frac{1}{2} e^x (\sin x + \cos x)$$

$$y=0 \text{ when } x=0$$

$$\therefore 1 = k - \frac{1}{2}$$

$$k = \frac{3}{2}$$

$$\therefore e^{-y} = \frac{3}{2} - \frac{1}{2} e^x (\sin x + \cos x)$$

$$[2e^{-y} = 3 - e^x (\sin x + \cos x)]$$

$$4 \quad \mathbf{a} \quad \frac{dN}{dt} = kN$$

$$\mathbf{b} \quad \int \frac{1}{N} dN = \int k dt$$

$$\ln|N| = kt + c$$

$$N = e^{kt+c} = e^c \times e^{kt}$$

$$N = Ae^{kt}$$

$$\mathbf{c} \quad t=0, N=40 \quad \therefore A=40$$

$$t=5, N=60 \quad \therefore 60 = 40e^{5k}$$

$$\therefore k = \frac{1}{5} \ln \frac{3}{2} = 0.0811 \text{ (3sf)}$$

$$\mathbf{d} \quad t=12 \quad \therefore N = 40e^{0.08109 \times 12}$$

$$= 106 \text{ (3sf)}$$

- 5 a let side length be l

$$A = 6l^2 \quad \therefore l = \sqrt{\frac{A}{6}}$$

$$V = l^3 = \left(\sqrt{\frac{A}{6}}\right)^3 = 6^{-\frac{3}{2}} A^{\frac{3}{2}}$$

$$\therefore \frac{dV}{dA} = \frac{3}{2} \times 6^{-\frac{3}{2}} \times A^{\frac{1}{2}} = k\sqrt{A}$$

b $\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$

$$\frac{dV}{dt} = cA$$

$$\therefore cA = k\sqrt{A} \times \frac{dA}{dt}$$

$$\frac{dA}{dt} = d\sqrt{A}$$

$$A = 100, \frac{dA}{dt} = 5 \quad \therefore d = \frac{1}{2}$$

$$\frac{dA}{dt} = \frac{1}{2}\sqrt{A}$$

$$\int 2A^{-\frac{1}{2}} dA = \int dt$$

$$4A^{\frac{1}{2}} = t + C$$

$$t = 10, A = 100 \quad \therefore C = 30$$

$$A^{\frac{1}{2}} = \frac{1}{4}(t + 30)$$

$$A = \frac{1}{16}(t + 30)^2$$

- 7 a $-\frac{dP}{dt} = k\sqrt{P}$

$$\int P^{-\frac{1}{2}} dP = \int -k dt$$

$$2P^{\frac{1}{2}} = -kt + c$$

$$\sqrt{P} = \frac{1}{2}c - \frac{1}{2}kt = a - bt$$

$$\therefore P = (a - bt)^2$$

- b $t = 0, P = 400 \quad \therefore \sqrt{400} = a - 0$

$$a = 20$$

- c $t = 30, P = 100 \quad \therefore \sqrt{100} = 20 - 30b$

$$b = \frac{1}{3}$$

$$\therefore P = \left(20 - \frac{1}{3}t\right)^2$$

$$t = 50 \quad \therefore P = \left(20 - \frac{50}{3}\right)^2$$

$$= 11\frac{1}{9}$$

- 6 a $-\frac{dm}{dt} = km$

$$\int \frac{1}{m} dm = \int -k dt$$

$$\ln |m| = -kt + c$$

$$m = e^{-kt+c} = e^c \times e^{-kt}$$

$$m = Ae^{-kt}$$

$$t = 0, m = 24 \quad \therefore A = 24$$

$$m = 24e^{-kt}$$

- b $t = 20, m = 22.6 \quad \therefore 22.6 = 24e^{-20k}$

$$\therefore k = -\frac{1}{20} \ln \frac{22.6}{24} = 0.00301 \text{ (3sf)}$$

- c $\frac{dm}{dt} = -km = -0.003005 \times 22.6$

$$= -0.0679 \text{ (3sf)}$$

\therefore decreasing at 0.0679 grams per day

- d $m = 12 \quad \therefore 12 = 24e^{-0.003005t}$

$$t = -\frac{1}{0.003005} \ln \frac{1}{2}$$

$$= 231 \text{ days (nearest day)}$$

- 8 a $-\frac{dV}{dt} = aV$ where a is a positive constant

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

if angle at vertex = 2θ , $\tan \theta = \frac{r}{h}$

$\therefore r = bh$ where b is a positive constant

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi b^2 h^3 \quad \therefore \frac{dV}{dh} = \pi b^2 h^2$$

$$\therefore -\pi b^2 h^2 \times \frac{dh}{dt} = a \times \frac{1}{3}\pi b^2 h^3$$

$$\frac{dh}{dt} = -kh \text{ where } k \text{ is a positive constant}$$

- b $\int \frac{1}{h} dh = \int -k dt$

$$\ln |h| = -kt + c$$

$$h = e^{-kt+c} = e^c \times e^{-kt} = Ae^{-kt}$$

$$t = 0, h = 12 \quad \therefore A = 12$$

$$t = 20, h = 10 \quad \therefore 10 = 12e^{-20k}$$

$$\therefore k = -\frac{1}{20} \ln \frac{5}{6}$$

$$\therefore h = 12e^{-kt}, k = 0.00912 \text{ (3sf)}$$

- c $6 = 12e^{-0.009116t}$

$$t = -\frac{1}{0.009116} \ln \frac{1}{2} = 76.0 \text{ (3sf)}$$

- 9 a** $\frac{1}{(1+x)(1-x)} \equiv \frac{A}{1+x} + \frac{B}{1-x}$
 $1 \equiv A(1-x) + B(1+x)$
 $x = -1 \Rightarrow A = \frac{1}{2}, x = 1 \Rightarrow B = \frac{1}{2}$
 $\frac{1}{(1+x)(1-x)} \equiv \frac{1}{2(1+x)} + \frac{1}{2(1-x)}$
- b** $t = 0, m = 0, \frac{dm}{dt} = 0.5$
 $\therefore 0.5 = k \times 1$
 $k = 0.5$
- c** $\int \frac{1}{(1+m)(1-m)} dm = \int \frac{1}{2} e^{-t} dt$
 $\int \left(\frac{\frac{1}{2}}{1+m} + \frac{\frac{1}{2}}{1-m} \right) dm = \int \frac{1}{2} e^{-t} dt$
 $\frac{1}{2} \ln |1+m| - \frac{1}{2} \ln |1-m| = -\frac{1}{2} e^{-t} + c$
 $\ln |1+m| - \ln |1-m| = C - e^{-t}$
 $t = 0, m = 0 \therefore 0 - 0 = C - 1$
 $C = 1$
 $\ln |1+m| - \ln |1-m| = 1 - e^{-t}$
 for $0 \leq m < 1$, $1+m > 0$ and $1-m > 0$
 $\therefore \ln(1+m) - \ln(1-m) = 1 - e^{-t}$
 $\ln \left(\frac{1+m}{1-m} \right) = 1 - e^{-t}$
- d** $m = 0.1 \therefore \ln \frac{1.1}{0.9} = 1 - e^{-t}$
 $t = -\ln \left(1 - \ln \frac{1.1}{0.9} \right) = 0.2240 \text{ hrs}$
 $= 13.4 \text{ minutes}$
- e** $t \rightarrow \infty, \ln \left(\frac{1+m}{1-m} \right) \rightarrow 1$
 \therefore limiting value of m is given by
 $\frac{1+m}{1-m} = e$
 $1+m = e(1-m)$
 $m(1+e) = e-1$
 $m = \frac{e-1}{1+e} = 0.4621$
 \therefore max. produced $\approx 462 \text{ g}$