INTEGRATION — Answers

1 $u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = \cos x,\ v = \sin x$

$\int x\cos x\ dx = x\sin x - \int \sin x\ dx$

$= x\sin x + \cos x + c$

2 **a** $u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = e^x,\ v = e^x$

$\int xe^x\ dx = xe^x - \int e^x\ dx$

$= xe^x - e^x + c$

$= e^x(x - 1) + c$

b $u = 4x,\ \frac{du}{dx} = 4;\ \frac{dv}{dx} = \sin x,\ v = -\cos x$

$\int 4x\sin x\ dx = -4x\cos x - \int -4\cos x\ dx$

$= -4x\cos x + \int 4\cos x\ dx$

$= -4x\cos x + 4\sin x + c$

c $u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = \cos 2x,\ v = \frac{1}{2}\sin 2x$

$\int x\cos 2x\ dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\ dx$

$= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$

d $u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = (x+1)^{\frac{1}{2}},\ v = \frac{2}{3}(x+1)^{\frac{3}{2}}$

$\int x\sqrt{x+1}\ dx = \frac{2}{3}x(x+1)^{\frac{3}{2}} - \int \frac{2}{3}(x+1)^{\frac{3}{2}}\ dx$

$= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + c$

$= \frac{2}{15}(x+1)^{\frac{3}{2}}[5x - 2(x + 1)] + c$

$= \frac{2}{15}(3x - 2)(x + 1)^{\frac{3}{2}} + c$

e $u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = e^{-3x},\ v = -\frac{1}{3}e^{-3x}$

$\int \frac{x}{e^{3x}}\ dx = -\frac{1}{3}xe^{-3x} - \int -\frac{1}{3}e^{-3x}\ dx$

$= -\frac{1}{3}xe^{-3x} + \int \frac{1}{3}e^{-3x}\ dx$

$= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + c$

$= -\frac{1}{9}e^{-3x}(3x + 1) + c$

f $u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = \sec^2 x,\ v = \tan x$

$\int x\sec^2 x\ dx = x\tan x - \int \tan x\ dx$

$= x\tan x + \int \frac{-\sin x}{\cos x}\ dx$

$= x\tan x + \ln|\cos x| + c$

3 **i** $u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = (2x + 1)^3,\ v = \frac{1}{8}(2x + 1)^4$

$\int x(2x + 1)^3\ dx = \frac{1}{8}x(2x + 1)^4 - \int \frac{1}{8}(2x + 1)^4\ dx$

$= \frac{1}{8}x(2x + 1)^4 - \frac{1}{80}(2x + 1)^5 + c$

$= \frac{1}{80}(2x + 1)^4[10x - (2x + 1)] + c$

$= \frac{1}{80}(8x - 1)(2x + 1)^4 + c$

ii $u = 2x + 1\ \therefore x = \frac{1}{2}(u - 1),\ \frac{du}{dx} = 2$

$\int x(2x + 1)^3\ dx = \int \frac{1}{2}(u - 1)u^3 \times \frac{1}{2}\ du$

$= \frac{1}{4}\int (u^4 - u^3)\ du$

$= \frac{1}{4}(\frac{1}{5}u^5 - \frac{1}{4}u^4) + c$

$= \frac{1}{4}[\frac{1}{5}(2x + 1)^5 - \frac{1}{4}(2x + 1)^4] + c$

$= \frac{1}{80}(2x + 1)^4[4(2x + 1) - 5] + c$

$= \frac{1}{80}(8x - 1)(2x + 1)^4 + c$, as for part **i**

4 $u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = e^{-x},\ v = -e^{-x}$

$$\int_0^2 xe^{-x}\ dx = [-xe^{-x}]_0^2 - \int_0^2 -e^{-x}\ dx = [-xe^{-x}]_0^2 + \int_0^2 e^{-x}\ dx$$
$$= [-xe^{-x} - e^{-x}]_0^2 = (-2e^{-2} - e^{-2}) - (0 - 1)$$
$$= 1 - 3e^{-2}$$

5 **a** $u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = \cos x,\ v = \sin x$

$$\int_0^{\frac{\pi}{6}} x\cos x\,dx = [x\sin x]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin x\,dx$$
$$= [x\sin x + \cos x]_0^{\frac{\pi}{6}}$$
$$= (\tfrac{\pi}{12} + \tfrac{\sqrt{3}}{2}) - (0 + 1)$$
$$= \tfrac{1}{12}(\pi + 6\sqrt{3} - 12)$$

b $u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = e^{2x},\ v = \frac{1}{2}e^{2x}$

$$\int_0^1 xe^{2x}\ dx = [\tfrac{1}{2}xe^{2x}]_0^1 - \int_0^1 \tfrac{1}{2}e^{2x}\ dx$$
$$= [\tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x}]_0^1$$
$$= (\tfrac{1}{2}e^2 - \tfrac{1}{4}e^2) - (0 - \tfrac{1}{4})$$
$$= \tfrac{1}{4}(e^2 + 1)$$

c $u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = \sin 3x,\ v = -\frac{1}{3}\cos 3x$

$$\int_0^{\frac{\pi}{4}} x\sin 3x\ dx = [-\tfrac{1}{3}x\cos 3x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} -\tfrac{1}{3}\cos 3x\ dx$$
$$= [-\tfrac{1}{3}x\cos 3x]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \tfrac{1}{3}\cos 3x\ dx$$
$$= [-\tfrac{1}{3}x\cos 3x + \tfrac{1}{9}\sin 3x]_0^{\frac{\pi}{4}}$$
$$= [-\tfrac{\pi}{12}(-\tfrac{1}{\sqrt{2}}) + \tfrac{1}{9}(\tfrac{1}{\sqrt{2}})] - (0)$$
$$= \tfrac{1}{72}\sqrt{2}\,(3\pi + 4)$$

6 **a** $u = x^2,\ \frac{du}{dx} = 2x;\ \frac{dv}{dx} = e^x,\ v = e^x$

$$\int x^2e^x\ dx = x^2e^x - \int 2xe^x\ dx$$

for $\int 2xe^x\ dx$, $u = 2x,\ \frac{du}{dx} = 2;\ \frac{dv}{dx} = e^x,\ v = e^x$

$$\int 2xe^x\ dx = 2xe^x - \int 2e^x\ dx$$
$$= 2xe^x - 2e^x + c$$

$$\therefore \int x^2e^x\ dx = x^2e^x - (2xe^x - 2e^x) + c$$
$$= e^x(x^2 - 2x + 2) + c$$

b $u = e^x,\ \frac{du}{dx} = e^x;\ \frac{dv}{dx} = \sin x,\ v = -\cos x$

$$\int e^x\sin x\ dx = -e^x\cos x - \int -e^x\cos x\ dx$$
$$= -e^x\cos x + \int e^x\cos x\ dx$$

for $\int e^x\cos x\ dx$, $u = e^x,\ \frac{du}{dx} = e^x;\ \frac{dv}{dx} = \cos x,\ v = \sin x$

$$\int e^x\cos x\ dx = e^x\sin x - \int e^x\sin x\ dx$$

$$\therefore \int e^x\sin x\ dx = -e^x\cos x + e^x\sin x - \int e^x\sin x\ dx$$
$$2\int e^x\sin x\ dx = -e^x\cos x + e^x\sin x + c$$
$$\int e^x\sin x\ dx = \tfrac{1}{2}e^x(\sin x - \cos x) + c$$

7 **a** $u = x^2,\ \frac{du}{dx} = 2x;\ \frac{dv}{dx} = \sin x,\ v = -\cos x$

$\int x^2 \sin x\ dx = -x^2 \cos x - \int -2x \cos x\ dx$

$= -x^2 \cos x + \int 2x \cos x\ dx$

for $\int 2x \cos x\ dx$, $u = 2x,\ \frac{du}{dx} = 2;\ \frac{dv}{dx} = \cos x,\ v = \sin x$

$\int 2x \cos x\ dx = 2x \sin x - \int 2 \sin x\ dx$

$= 2x \sin x + 2 \cos x + c$

$\therefore \int x^2 \sin x\ dx = -x^2 \cos x + (2x \sin x + 2 \cos x) + c$

$= (2 - x^2)\cos x + 2x \sin x + c$

b $u = x^2,\ \frac{du}{dx} = 2x;\ \frac{dv}{dx} = e^{3x},\ v = \frac{1}{3}e^{3x}$

$\int x^2 e^{3x}\ dx = \frac{1}{3}x^2 e^{3x} - \int \frac{2}{3}xe^{3x}\ dx$

for $\int \frac{2}{3}xe^{3x}\ dx$, $u = \frac{2}{3}x,\ \frac{du}{dx} = \frac{2}{3};\ \frac{dv}{dx} = e^{3x},\ v = \frac{1}{3}e^{3x}$

$\int \frac{2}{3}xe^{3x}\ dx = \frac{2}{9}xe^{3x} - \int \frac{2}{9}e^{3x}\ dx$

$= \frac{2}{9}xe^{3x} - \frac{2}{27}e^{3x} + c$

$\therefore \int x^2 e^{3x}\ dx = \frac{1}{3}x^2 e^{3x} - (\frac{2}{9}xe^{3x} - \frac{2}{27}e^{3x}) + c$

$= \frac{1}{27}e^{3x}(9x^2 - 6x + 2) + c$

c $u = e^{-x},\ \frac{du}{dx} = -e^{-x};\ \frac{dv}{dx} = \cos 2x,\ v = \frac{1}{2}\sin 2x$

$\int e^{-x} \cos 2x\ dx = \frac{1}{2}e^{-x} \sin 2x - \int -\frac{1}{2}e^{-x} \sin 2x\ dx$

$= \frac{1}{2}e^{-x} \sin 2x + \int \frac{1}{2}e^{-x} \sin 2x\ dx$

for $\int \frac{1}{2}e^{-x} \sin 2x\ dx$, $u = \frac{1}{2}e^{-x},\ \frac{du}{dx} = -\frac{1}{2}e^{-x};\ \frac{dv}{dx} = \sin 2x,\ v = -\frac{1}{2}\cos 2x$

$\int \frac{1}{2}e^{-x} \sin 2x\ dx = -\frac{1}{4}e^{-x} \cos 2x - \int \frac{1}{4}e^{-x} \cos 2x\ dx$

$\therefore \int e^{-x} \cos 2x\ dx = \frac{1}{2}e^{-x} \sin 2x - \frac{1}{4}e^{-x} \cos 2x - \frac{1}{4}\int e^{-x} \cos 2x\ dx$

$\frac{5}{4}\int e^{-x} \cos 2x\ dx = \frac{1}{2}e^{-x} \sin 2x - \frac{1}{4}e^{-x} \cos 2x + c$

$\int e^{-x} \cos 2x\ dx = \frac{1}{5}e^{-x}(2 \sin 2x - \cos 2x) + c$

8 **a** $\frac{1}{x}$

b $u = \ln x,\ \frac{du}{dx} = \frac{1}{x};\ \frac{dv}{dx} = 1,\ v = x$

$\int \ln x\ dx = x \ln x - \int \frac{1}{x} \times x\ dx$

$= x \ln x - \int dx$

$= x \ln x - x + c$

$= x(\ln x - 1) + c$

9 **a** $u = \ln 2x$, $\frac{du}{dx} = \frac{1}{x}$; $\frac{dv}{dx} = 1$, $v = x$

$\int \ln 2x \, dx = x \ln 2x - \int \frac{1}{x} \times x \, dx$

$= x \ln 2x - \int dx$

$= x \ln 2x - x + c$

$= x(\ln 2x - 1) + c$

b $u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$; $\frac{dv}{dx} = 3x$, $v = \frac{3}{2}x^2$

$\int 3x \ln x \, dx = \frac{3}{2}x^2 \ln x - \int \frac{1}{x} \times \frac{3}{2}x^2 \, dx$

$= \frac{3}{2}x^2 \ln x - \int \frac{3}{2}x \, dx$

$= \frac{3}{2}x^2 \ln x - \frac{3}{4}x^2 + c$

$= \frac{3}{4}x^2(2 \ln x - 1) + c$

c $u = (\ln x)^2$, $\frac{du}{dx} = 2(\ln x) \times \frac{1}{x}$; $\frac{dv}{dx} = 1$, $v = x$

$\int (\ln x)^2 \, dx = x(\ln x)^2 - \int 2 \ln x \, dx$

for $\int 2 \ln x \, dx$, $u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$; $\frac{dv}{dx} = 2$, $v = 2x$

$\int 2 \ln x \, dx = 2x \ln x - \int 2 \, dx$

$= 2x \ln x - 2x + c$

$\therefore \int (\ln x)^2 \, dx = x(\ln x)^2 - (2x \ln x - 2x) + c$

$= x[(\ln x)^2 - 2 \ln x + 2] + c$

10 **a** $u = x + 2$, $\frac{du}{dx} = 1$; $\frac{dv}{dx} = e^x$, $v = e^x$

$\int_{-1}^{0} (x + 2)e^x \, dx$

$= [(x + 2)e^x]_{-1}^{0} - \int_{-1}^{0} e^x \, dx$

$= [(x + 2)e^x - e^x]_{-1}^{0}$

$= (2 - 1) - (e^{-1} - e^{-1})$

$= 1$

b $u = \ln x$, $\frac{du}{dx} = \frac{1}{x}$; $\frac{dv}{dx} = x^2$, $v = \frac{1}{3}x^3$

$\int_{1}^{2} x^2 \ln x \, dx$

$= [\frac{1}{3}x^3 \ln x]_{1}^{2} - \int_{1}^{2} \frac{1}{3}x^2 \, dx$

$= [\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3]_{1}^{2}$

$= (\frac{8}{3} \ln 2 - \frac{8}{9}) - (0 - \frac{1}{9})$

$= \frac{8}{3} \ln 2 - \frac{7}{9}$

c $u = 2x$, $\frac{du}{dx} = 2$; $\frac{dv}{dx} = e^{3x-1}$, $v = \frac{1}{3}e^{3x-1}$

$\int_{\frac{1}{3}}^{1} 2xe^{3x-1} \, dx$

$= [\frac{2}{3}xe^{3x-1}]_{\frac{1}{3}}^{1} - \int_{\frac{1}{3}}^{1} \frac{2}{3}e^{3x-1} \, dx$

$= [\frac{2}{3}xe^{3x-1} - \frac{2}{9}e^{3x-1}]_{\frac{1}{3}}^{1}$

$= (\frac{2}{3}e^2 - \frac{2}{9}e^2) - (\frac{2}{9} - \frac{2}{9})$

$= \frac{4}{9}e^2$

d $u = \ln(2x + 3)$, $\frac{du}{dx} = \frac{2}{2x+3}$; $\frac{dv}{dx} = 1$, $v = x$

$\int_{0}^{3} \ln(2x + 3) \, dx$

$= [x \ln(2x + 3)]_{0}^{3} - \int_{0}^{3} \frac{2x}{2x+3} \, dx$

$= [x \ln(2x + 3)]_{0}^{3} - \int_{0}^{3} \frac{(2x+3)-3}{2x+3} \, dx$

$= [x \ln(2x + 3)]_{0}^{3} - \int_{0}^{3} (1 - \frac{3}{2x+3}) \, dx$

$= [x \ln(2x + 3) - x + \frac{3}{2} \ln|2x + 3|]_{0}^{3}$

$= (3 \ln 9 - 3 + \frac{3}{2} \ln 9) - (0 - 0 + \frac{3}{2} \ln 3)$

$= \frac{15}{2} \ln 3 - 3$

e $u = x^2$, $\frac{du}{dx} = 2x$; $\frac{dv}{dx} = \cos x$, $v = \sin x$

$\int x^2 \cos x \; dx = x^2 \sin x - \int 2x \sin x \; dx$

for $\int 2x \sin x \; dx$, $u = 2x$, $\frac{du}{dx} = 2$; $\frac{dv}{dx} = \sin x$, $v = -\cos x$

$$\int 2x \sin x \; dx = -2x \cos x - \int -2\cos x \; dx$$
$$= -2x \cos x + \int 2\cos x \; dx$$
$$= -2x \cos x + 2 \sin x + c$$

$$\therefore \int_0^{\frac{\pi}{2}} x^2 \cos x \; dx = [x^2 \sin x + 2x \cos x - 2 \sin x]_0^{\frac{\pi}{2}}$$
$$= (\tfrac{1}{4}\pi^2 + 0 - 2) - (0 + 0 - 0)$$
$$= \tfrac{1}{4}\pi^2 - 2$$

f $u = e^{3x}$, $\frac{du}{dx} = 3e^{3x}$; $\frac{dv}{dx} = \sin 2x$, $v = -\frac{1}{2}\cos 2x$

$$\int e^{3x} \sin 2x \; dx = -\tfrac{1}{2}e^{3x} \cos 2x - \int -\tfrac{3}{2}e^{3x} \cos 2x \; dx$$
$$= -\tfrac{1}{2}e^{3x} \cos 2x + \int \tfrac{3}{2}e^{3x} \cos 2x \; dx$$

for $\int \frac{3}{2}e^{3x} \cos 2x \; dx$, $u = \frac{3}{2}e^{3x}$, $\frac{du}{dx} = \frac{9}{2}e^{3x}$; $\frac{dv}{dx} = \cos 2x$, $v = \frac{1}{2}\sin 2x$

$$\int \tfrac{3}{2}e^{3x} \cos 2x \; dx = \tfrac{3}{4}e^{3x} \sin 2x - \int \tfrac{9}{4}e^{3x} \sin 2x \; dx$$

$$\therefore \int e^{3x} \sin 2x \; dx = -\tfrac{1}{2}e^{3x} \cos 2x + \tfrac{3}{4}e^{3x} \sin 2x - \int \tfrac{9}{4}e^{3x} \sin 2x \; dx$$

$$\tfrac{13}{4}\int e^{3x} \sin 2x \; dx = -\tfrac{1}{2}e^{3x} \cos 2x + \tfrac{3}{4}e^{3x} \sin 2x + c$$

$$\therefore \int_0^{\frac{\pi}{4}} e^{3x} \sin 2x \; dx = \tfrac{4}{13}[-\tfrac{1}{2}e^{3x} \cos 2x + \tfrac{3}{4}e^{3x} \sin 2x]_0^{\frac{\pi}{4}}$$
$$= \tfrac{4}{13}[(0 + \tfrac{3}{4}e^{\frac{3\pi}{4}}) - (-\tfrac{1}{2} + 0)]$$
$$= \tfrac{1}{13}(3e^{\frac{3\pi}{4}} + 2)$$