

# INTEGRATION

# Answers

1    **a** =  $2 \sin x + c$       **b** =  $-\frac{1}{4} \cos 4x + c$       **c** =  $2 \sin \frac{1}{2}x + c$       **d** =  $-\cos(x + \frac{\pi}{4}) + c$   
**e** =  $\frac{1}{2} \sin(2x - 1) + c$     **f** =  $3 \cos(\frac{\pi}{3} - x) + c$     **g** =  $\sec x + c$       **h** =  $-\cot x + c$   
**i** =  $\frac{5}{2} \tan 2x + c$       **j** =  $-4 \operatorname{cosec} \frac{1}{4}x + c$     **k** =  $\int 4 \operatorname{cosec}^2 x \, dx$     **l** =  $\int \sec^2(4x + 1) \, dx$   
 $= -4 \cot x + c$        $= \frac{1}{4} \tan(4x + 1) + c$

2    **a** =  $[\sin x]_0^{\frac{\pi}{2}}$       **b** =  $[-\frac{1}{2} \cos 2x]_0^{\frac{\pi}{6}}$       **c** =  $[4 \sec \frac{1}{2}x]_0^{\frac{\pi}{2}}$   
 $= 1 - 0 = 1$        $= -\frac{1}{4} - (-\frac{1}{2}) = \frac{1}{4}$        $= 4\sqrt{2} - 4 = 4(\sqrt{2} - 1)$   
**d** =  $[\frac{1}{2} \sin(2x - \frac{\pi}{3})]_0^{\frac{\pi}{3}}$     **e** =  $[\frac{1}{3} \tan 3x]_0^{\frac{\pi}{4}}$       **f** =  $[-\operatorname{cosec} x]_{\frac{\pi}{3}}^{\frac{2\pi}{3}}$   
 $= \frac{\sqrt{3}}{4} - (-\frac{\sqrt{3}}{4}) = \frac{\sqrt{3}}{2}$        $= 0 - (-\frac{1}{3}) = \frac{1}{3}$        $= -\frac{2}{\sqrt{3}} - (-1) = 1 - \frac{2}{3}\sqrt{3}$

3    **a**  $\tan^2 \theta = \sec^2 \theta - 1$   
**b**  $\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + c$

4    **a**  $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$   
let  $B = A \Rightarrow \cos 2A \equiv \cos^2 A - \sin^2 A$   
 $\cos 2A \equiv \cos^2 A - (1 - \cos^2 A)$   
 $\cos 2A \equiv 2 \cos^2 A - 1$   
 $\cos^2 A \equiv \frac{1}{2}(1 + \cos 2A)$

**b**  $\int \cos^2 x \, dx = \int (\frac{1}{2} + \frac{1}{2} \cos 2x) \, dx = \frac{1}{2}x + \frac{1}{4} \sin 2x + c$

5    **a** =  $\int (\frac{1}{2} - \frac{1}{2} \cos 2x) \, dx$       **b** =  $\int (\operatorname{cosec}^2 2x - 1) \, dx$   
 $= \frac{1}{2}x - \frac{1}{4} \sin 2x + c$        $= -\frac{1}{2} \cot 2x - x + c$

**c** =  $\int \frac{1}{2} \sin 2x \, dx$       **d** =  $\int \frac{1}{\cos x} \times \frac{\sin x}{\cos x} \, dx$   
 $= -\frac{1}{4} \cos 2x + c$        $= \int \sec x \tan x \, dx$   
 $= \sec x + c$

**e** =  $\int (2 + 2 \cos 6x) \, dx$       **f** =  $\int (1 + 2 \sin x + \sin^2 x) \, dx$   
 $= 2x + \frac{1}{3} \sin 6x + c$        $= \int (1 + 2 \sin x + \frac{1}{2} - \frac{1}{2} \cos 2x) \, dx$   
 $= \int (\frac{3}{2} + 2 \sin x - \frac{1}{2} \cos 2x) \, dx$   
 $= \frac{3}{2}x - 2 \cos x - \frac{1}{4} \sin 2x + c$

**g** =  $\int (\sec^2 x - 2 \sec x \tan x + \tan^2 x) \, dx$     **h** =  $\int \frac{1}{\sin 2x} \times \frac{\cos x}{\sin x} \, dx$   
 $= \int (\sec^2 x - 2 \sec x \tan x + \sec^2 x - 1) \, dx$      $= \int \frac{1}{2 \sin x \cos x} \times \frac{\cos x}{\sin x} \, dx$   
 $= \int (2 \sec^2 x - 2 \sec x \tan x - 1) \, dx$      $= \int \frac{1}{2} \operatorname{cosec}^2 x \, dx$   
 $= 2 \tan x - 2 \sec x - x + c$        $= -\frac{1}{2} \cot x + c$

$$\begin{aligned}
 \mathbf{i} &= \int (\cos^2 x)^2 dx \\
 &= \int \left[\frac{1}{2}(1 + \cos 2x)\right]^2 dx \\
 &= \frac{1}{4} \int (1 + 2 \cos 2x + \cos^2 2x) dx \\
 &= \frac{1}{4} \int \left[1 + 2 \cos 2x + \frac{1}{2}(1 + \cos 4x)\right] dx \\
 &= \frac{1}{8} \int (3 + 4 \cos 2x + \cos 4x) dx \\
 &= \frac{1}{8} (3x + 2 \sin 2x + \frac{1}{4} \sin 4x) + c \\
 &= \frac{3}{8}x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} &= \int_0^{\frac{\pi}{2}} (1 + \cos 2x) dx \\
 &= \left[x + \frac{1}{2} \sin 2x\right]_0^{\frac{\pi}{2}} \\
 &= \left(\frac{\pi}{2} + 0\right) - (0 + 0) = \frac{\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} &= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sec^2 \frac{1}{2}x - 1) dx \\
 &= \left[2 \tan \frac{1}{2}x - x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
 &= \left(2 - \frac{\pi}{2}\right) - \left(\frac{2}{\sqrt{3}} - \frac{\pi}{3}\right) \\
 &= 2 - \frac{2}{3}\sqrt{3} - \frac{\pi}{6}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} &= \int_0^{\frac{\pi}{4}} (1 - 4 \sin x + 4 \sin^2 x) dx \\
 &= \int_0^{\frac{\pi}{4}} [1 - 4 \sin x + 2(1 - \cos 2x)] dx \\
 &= \int_0^{\frac{\pi}{4}} (3 - 4 \sin x - 2 \cos 2x) dx \\
 &= \left[3x + 4 \cos x - \sin 2x\right]_0^{\frac{\pi}{4}} \\
 &= \left(\frac{3\pi}{4} + 2\sqrt{2} - 1\right) - (0 + 4 - 0) \\
 &= \frac{3\pi}{4} + 2\sqrt{2} - 5
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{7} \quad \mathbf{a} \quad \sin(A + B) &\equiv \sin A \cos B + \cos A \sin B \quad (1) \\
 \sin(A - B) &\equiv \sin A \cos B - \cos A \sin B \quad (2) \\
 (1) + (2) &\Rightarrow \sin(A + B) + \sin(A - B) \equiv 2 \sin A \cos B \\
 \sin A \cos B &\equiv \frac{1}{2} [\sin(A + B) + \sin(A - B)]
 \end{aligned}$$

$$\mathbf{b} = \int \left(\frac{1}{2} \sin 4x + \frac{1}{2} \sin 2x\right) dx = -\frac{1}{8} \cos 4x - \frac{1}{4} \cos 2x + c$$

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} &= \int (\cos 4x - \cos 6x) dx \\
 &= \frac{1}{4} \sin 4x - \frac{1}{6} \sin 6x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} &= \int [2 \sin 5x + 2 \sin(-3x)] dx \\
 &= \int (2 \sin 5x - 2 \sin 3x) dx \\
 &= -\frac{2}{5} \cos 5x + \frac{2}{3} \cos 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} &= \int_0^{\frac{\pi}{4}} \frac{1}{2} \sin 4x dx \\
 &= \left[-\frac{1}{8} \cos 4x\right]_0^{\frac{\pi}{4}} \\
 &= \frac{1}{8} - \left(-\frac{1}{8}\right) = \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sin 2x} \times \frac{\cos 2x}{\sin 2x} dx \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} 2x \cot 2x dx \\
 &= \left[-\frac{1}{2} \operatorname{cosec} 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \\
 &= -\frac{1}{2} - \left(-\frac{1}{\sqrt{3}}\right) = \frac{1}{3}\sqrt{3} - \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x} \times \frac{1}{\sin^2 x} dx \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{(\frac{1}{2} \sin 2x)^2} dx \\
 &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4 \operatorname{cosec}^2 2x dx \\
 &= \left[-2 \cot 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
 &= \frac{2}{\sqrt{3}} - \left(-\frac{2}{\sqrt{3}}\right) \\
 &= \frac{4}{3}\sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} &= \int \left(\frac{1}{2} \cos 3x + \frac{1}{2} \cos x\right) dx \\
 &= \frac{1}{6} \sin 3x + \frac{1}{2} \sin x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} &= \int \left[\frac{1}{2} \sin\left(2x + \frac{\pi}{6}\right) - \frac{1}{2} \sin \frac{\pi}{6}\right] dx \\
 &= \int \left[\frac{1}{2} \sin\left(2x + \frac{\pi}{6}\right) - \frac{1}{4}\right] dx \\
 &= -\frac{1}{4} \cos\left(2x + \frac{\pi}{6}\right) - \frac{1}{4}x + c
 \end{aligned}$$