1. Find $\int x \cos 3x \, dx$.

[4]

2. Find $\int x^8 \ln(3x) \, dx$.

[5]

3.
   i. Use division to show that $\dfrac{t^3}{t+2} \equiv t^2 - 2t + 4 - \dfrac{8}{t+2}$.

   [3]

   ii. Find $\displaystyle\int_1^2 6t^2 \ln(t+2) \, dt$. Give your answer in the form $A + B \ln 3 + C \ln 4$.

   [6]

4. Find the exact value of $\displaystyle\int_1^8 \dfrac{1}{\sqrt[3]{x}} \ln x \, dx$, giving your answer in the form $A \ln 2 + B$, where $A$ and $B$ are constants to be found.

[5]

5. Find $\displaystyle\int (2x+1) \ln x \, dx$.

[5]

6. Show that $\displaystyle\int_0^1 16x e^{4x} \, dx = 3e^4 + 1$.

[5]

END OF QUESTION paper

# Mark scheme

| Question | Answer/Indicative content | Marks | Part marks and guidance | |
|---|---|---|---|---|
| 1 | $u = x$ and $dv = \cos 3x$ | M1 | integration by parts as far as $f(x) \pm \int g(x)dx$ | Check if labelled $v, du$ |
| | $x \times \dfrac{1}{3}\sin 3x - \int \dfrac{1}{3}\sin 3x dx$ | A2 | **A1** for $x \times k\sin 3x - \int k\sin 3x\,dx$; $\quad k \neq \dfrac{1}{3}$ or 0 <br><br> Not ..... $\dfrac{1}{3}\left(\dfrac{1}{3}\cos 3x\right)$ or ..... $- -\dfrac{1}{9}\cos 3x$ | $k$ may be negative |
| | $\dfrac{x}{3}\sin 3x + \dfrac{1}{9}\cos 3x\,[\,+\,c\,]$ <br> cao www ISW | A1 | **Examiner's Comments** <br><br> The vast majority recognised this question as one suitable for integration by parts, the main errors arising from the integrations of $\cos 3x$ and $\sin 3x$. Provided the method of integrating by parts was fully understood, some credit was given to candidates who used a wrong sign or 3 instead of $\dfrac{1}{3}$ in the integrals. Candidates were expected to simplify $\dfrac{1}{3}\left(\dfrac{1}{3}\cos 3x\right)$ and $--\dfrac{1}{9}\cos 3x$ in their answers but, needless to say, they were not expected to multiply their result by 9 to make it look 'better'. | |
| | **Total** | **4** | | |
| 2 | $u = \ln 3x$ and $dv$ or $\dfrac{dv}{dx} = x^8$ | M1 | integ by parts as far as $f(x)+/- \int g(x)(dx)$ | If difficult to assess, $x^8$ must be integrated, so look for term in $x^9$ |
| | $\dfrac{d}{dx}(\ln 3x) = \dfrac{1}{x}$ or $\dfrac{3}{3x}$ | B1 | stated or clearly used | |
| | $\dfrac{x^9}{9}\ln 3x - \int \dfrac{x^9}{9}$ their $\dfrac{du}{dx}(dx)$  FT | √A1 | i.e. correct understanding of 'by parts'… | ..even if $\ln(3x)$ incorrectly differentiated |

Indication that $\int kx^8\mathrm{d}x$ is required

| | | |
|---|---|---|
| M1 | | i.e. before integrating, product of terms must be taken |

$\dfrac{x^9}{9}\ln 3x - \dfrac{x^9}{81}$ or $\dfrac{1}{9}x^9\left(\ln 3x - \dfrac{1}{9}\right)$ ISW (+c) <u>cao</u>

| | | |
|---|---|---|
| A1 | | $\dfrac{1}{9}\dfrac{x^9}{9}$ to be simplif to $\dfrac{x^9}{81}$ ; $\dfrac{3x^9}{243}$ satis |

If candidate manipulates ln(3$x$) first of all

ln(3$x$) = ln 3 + ln $x$

| | |
|---|---|
| B1 | |

If, however, ln(3$x$) is said to be ln 3.ln $x$, then B0 followed by possible M1 A1

$u = \ln x$ and $\mathrm{d}v = x^8$

| | | |
|---|---|---|
| M1 | | In order to find $\int x^8\ln x\,\mathrm{d}x$ : |

A1 in line with alternative solution on

$\dfrac{x^9}{9}\ln x - \int \dfrac{x^9}{9}\cdot\dfrac{1}{x}(\mathrm{d}x)$ or better

| | |
|---|---|
| A1 | |

$\dfrac{x^9}{9}\ln x - \dfrac{x^9}{81}$

| | |
|---|---|
| A1 | |

LHS, where the 'M' mark is for dealing with $\int x^8 \ln x\,\mathrm{d}x$ 'by parts' in the right order and the 2 @ A1 are for correct results.

| | |
|---|---|
| √A1 | |

Their $\int x^8 \ln x\,\mathrm{d}x + \dfrac{x^9}{9}\ln 3$ (+ c) FT ISW

**Examiner's Comments**

This was a relatively straightforward question but two specific errors occurred. The first could have been forecast: the differentiation of ln(3x) as $\dfrac{1}{3x}$; the other, perhaps not so predictable, involved the integration (at the second stage) of $\dfrac{x^9}{9}\cdot\dfrac{1}{x}$. Here examiners were looking, first of all, to see if candidates were integrating an expression of the form $kx^8$. Even the correct simplification at that stage was of ten followed by the incorrect result of $\dfrac{1}{72}x^9$. However, it can be said that the technique of 'integration by parts' was generally known.

| | | | | | | Integration by Parts |
|---|---|---|---|---|---|---|
| | | Total | | 5 | | |
| 3 | i | $t^2$ in quotient and $t^3 + 2t^2$ seen | B1 | | or $\dfrac{t(t^2-4)+4t}{(t+2)}$ | or $\dfrac{(t+2)^3 - 6t^2 - 12t - 8}{(t+2)}$ |
| | i | $-2t$ in quotient $-2t^2 - (-2t^2 - 4t) = 4t$ seen | B1 | | $\dfrac{t(t+2)(t-2)}{(t+2)} + \dfrac{4t}{t+2}$ | $\dfrac{(t+2)^3}{(t+2)} - \dfrac{6((t+2)^2 - 4t - 4) + 12t + 8}{(t+2)}$  oe |
| | | | | | | $(t+2)^2 - 6(t+2) + \dfrac{12t+16}{t+2}$  oe |
| | i | completion to obtain correct quotient and remainder identified www | B1 | | $t(t-2) + \dfrac{4(t+2)-8}{t+2}$ | or  B1 for $\dfrac{t^2(t+2)-2t^2}{(t+2)}$ |
| | | | | | | both steps needed for final B1 |
| | | alternatively | | | | |
| | i | $\dfrac{t^3}{t+2} \equiv At^2 + Bt + C + \dfrac{D}{(t+2)}$ | B1 | | or $t^3 \equiv (At^2 + Bt + C)(t+2) + D$ | or  B1 for $\dfrac{t^2(t+2)-2t^2}{(t+2)}$ |
| | | equate coefficients to obtain correctly | | | | B1 for $t^2 + \dfrac{-2t(t+2)+4t}{(t+2)}$ |
| | i | $A = 1$, $0 = 2A + B$ and $B = -2$ www | B1 | | | |
| | | | | | **Examiner's Comments** | |
| | i | $0 = 2B + C$ and $0 = 2C + D$ obtained and solved correctly www | B1 | | Most candidates took the expected route and showed the required result successfully using long division, although a proportion who adopted this approach made sign errors and fudged the rest. A variety of other approaches were also seen. Candidates are reminded that in this type of question, a convincing argument is required – it appeared that some strong candidates lost marks because the answer a andppeared obvious to them. | B1 for $t^2 - 2t + \dfrac{4(t+2)-8}{(t+2)}$ |

| | | | | | | |
|---|---|---|---|---|---|---|
| | ii | integration by parts with $u = \ln(t+2)$ and $dv = 6t^2$ to obtain $f(t) \pm \int g(t)(dt)$ | M1* | $f(t)$ must include $t^3$ and $g(t)$ must **not** include a logarithm | **Integration by Parts** <br> ignore spurious d$x$ etc |
| | ii | $2t^3 \ln(t+2) - \int \dfrac{2t^3}{t+2}(dt)$ cao | A1 | | alternatively, following $u = t+2$ |
| | ii | result from part (i) seen in integrand; must follow award of at least first M1 | M1* | no integration required for this mark | $\int 2(u^2 - 6u + 12 - \dfrac{8}{u})du$ oe |
| | ii | $F[t] = 2t^3 \ln(t+2) \pm \dfrac{2t^3}{3} \pm 2t^2 \pm 8t \pm 16\ln(t+2)$ | A1 | $2t^3 \ln(t+2) - \dfrac{2t^3}{3} + 2t^2 - 8t + 16\ln(t+2)$ | $\dfrac{2u^3}{3} - 6u^2 + 24u - 16\ln u$ <br> and <br> $2t^3 \ln(t+2)$ |
| | ii | their F[2] – F[1] | M1dep* | at least one of their terms correctly integrated | NB limits following substitution are $u = 4$ and $u = 3$ |
| | ii | $-6\tfrac{2}{3} - 18\ln3 + 32\ln4$ oe cao | A1 | | |
| | ii | | | **Examiner's Comments** <br><br> Most candidates made some progress here. Integration by parts was generally used, and mostly successfully. Weak candidates failed to make the connection with part (i), but those who did make the connection generally went on to achieve at least the method marks. It was often in the manipulation following integration that marks were lost. A surprisingly common error was  | |
| | | **Total** | **9** | | |
| 4 | | $Ax^{\frac{2}{3}} \ln x - \int Bx^{\frac{2}{3}} \times \dfrac{1}{x} dx$ oe | M1* | $A$ and $B$ are non-zero constants; | |

| | | |
|---|---|---|
| $\dfrac{3}{2}x^{\frac{2}{3}}\ln x - \displaystyle\int \dfrac{3}{2}x^{\frac{2}{3}}\times\dfrac{1}{x}\,\mathrm{d}x$ | A1 | ignore + *c* |

| | | |
|---|---|---|
| $F[x] = \dfrac{3}{2}x^{\frac{2}{3}}\ln x - \dfrac{3/2}{2/3}x^{\frac{2}{3}}$ | A1 | ignore limits for first three marks |
| F[8] − F[1] | M1*dep | and also dependent on integration of their $\quad$ their $\dfrac{3}{2}x^{-\frac{1}{3}}$ |

NB A0 for $6\ln 8 - \dfrac{27}{4}$

**Examiner's Comments**

Most candidates knew how to integrate by parts, but many made accuracy errors, particularly when dealing with the second integral. Some candidates worked carefully through the problem, but either didn't see the instruction to leave the answer in terms of ln2, or didn't know how to resolve ln8.

| | | |
|---|---|---|
| $18\ln 2 - \dfrac{27}{4}$ cao | A1 | |

| | | |
|---|---|---|
| **Total** | **5** | |

| | | | |
|---|---|---|---|
| 5 | $u = \ln x,\ \mathrm{d}v = 2x+1$ | M1(AO1.1a) | Recognise integration by parts with correct *u* and d*v* |
| | $\mathrm{d}u = \dfrac{1}{x},\ v = x^2 + x$ | B1(AO1.2) | State or imply that $\mathrm{d}u = \dfrac{1}{x}$ |
| | | A1(AO1.1) | |

| | | | | | | | Integration by Parts |
|---|---|---|---|---|---|---|---|
| | | $I = (x^2 + x)\ln x - \int (x^2 + x)\frac{1}{x}\,dx$ | M1(AO1.1a) | Correct unsimplified expression | | | |
| | | $= (x^2 + x)\ln x - \int (x + 1)\,dx$ | A1(AO1.1) | Attempt to simplify and integrate | | | |
| | | | [5] | | | | |
| | | $= (x^2 + x)\ln x - (\frac{1}{2}x^2 + x) + c$ | | Obtain fully correct integral | Including $+ c$ | | |

| | Total | 5 | | | |
|---|---|---|---|---|---|

| 6 | $\dfrac{1}{4}\mathrm{e}^{4x}$ soi<br><br>$[16]x \times \dfrac{1}{4}\mathrm{e}^{4x} - \int [16] \times \dfrac{1}{4}\mathrm{e}^{4x}\,dx$ oe<br><br>F[x] = [4xe$^{4x}$ – e$^{4x}$]<br><br>F[1] – F[0]<br><br><br>= 3e$^4$ + 1    NB AG | B1<br><br>M1*<br><br><br>A1<br><br>M1dep*<br><br><br><br>A1<br><br>[5] | from integration<br><br>allow sign errors only<br><br><br><br>allow bracket errors, but substitution of limits must be shown<br><br>convincing intermediate step needed eg $4\mathrm{e}^4 - \mathrm{e}^4 - (0 - \mathrm{e}^0)$ | <br><br>ignore limits at this stage<br><br><br><br>NB double negative may be implied by plus sign<br><br><br>no recovery from bracket errors for this mark | |
|---|---|---|---|---|---|

| | | | | | | Integration by Parts |
|---|---|---|---|---|---|---|---|
| | | | | | **Examiner's Comments**<br><br>This was question was done very well, with many candidates achieving full marks. A few candidates integrated x when applying integration by parts, and more often than not the correct result mysteriously appeared from wrong working. | |
| | | | Total | 5 | | | |