

1. The temperature of a freezer is  $-20^{\circ}\text{C}$ . A container of a liquid is placed in the freezer. The rate at which the temperature,  $\theta^{\circ}\text{C}$ , of a liquid decreases is proportional to the difference in temperature between the liquid and its surroundings. The situation is modelled by the differential equation

$$\frac{d\theta}{dt} = -k(\theta + 20),$$

where time  $t$  is in minutes and  $k$  is a positive constant.

- i. Express  $\theta$  in terms of  $t$ ,  $k$  and an arbitrary constant.

[3]

Initially the temperature of the liquid in the container is  $40^{\circ}\text{C}$  and, at this instant, the liquid is cooling at a rate of  $3^{\circ}\text{C}$  per minute. The liquid freezes at  $0^{\circ}\text{C}$ .

- ii. Find the value of  $k$  and find also the time it takes (to the nearest minute) for the liquid to freeze.

[5]

The procedure is repeated on another occasion with a different liquid. The initial temperature of this liquid is  $90^{\circ}\text{C}$ . After 19 minutes its temperature is  $0^{\circ}\text{C}$ .

- iii. Without any further calculation, explain what you can deduce about the value of  $k$  in this case.

[1]

2. At time  $t$  seconds, the radius of a spherical balloon is  $r$  cm. The balloon is being inflated so that the rate of increase of its radius is inversely proportional to the square root of its radius. When  $t = 5$ ,  $r = 9$  and, at this instant, the radius is increasing at  $1.08 \text{ cm s}^{-1}$ .

- i. Write down a differential equation to model this situation, and solve it to express  $r$  in terms of  $t$ .

[7]

- ii. How much air is in the balloon initially?

[2]

[The volume of a sphere is  $V = \frac{4}{3}\pi r^3$ .]

3. A container in the shape of an inverted cone of radius 3 metres and vertical height 4.5 metres is initially filled with liquid fertiliser. This fertiliser is released through a hole in the bottom of the container at a rate of  $0.01 \text{ m}^3$  per second. At time  $t$  seconds the fertiliser remaining in the container forms an inverted cone of height  $h$  metres.

[The volume of a cone is  $V = \frac{1}{3}\pi r^2 h$ .]

i. Show that  $h^2 \frac{dh}{dt} = -\frac{9}{400\pi}$ .

[5]

- ii. Express  $h$  in terms of  $t$ .

[4]

- iii. Find the time it takes to empty the container, giving your answer to the nearest minute.

[2]

4. In the year 2000 the population density,  $P$ , of a village was 100 people per  $\text{km}^2$ , and was increasing at the rate of 1 person per  $\text{km}^2$  per year. The rate of increase of the population density is thought to be inversely proportional to the size of the population density. The time in years after the year 2000 is denoted by  $t$ .

- i. Write down a differential equation to model this situation, and solve it to express  $P$  in terms of  $t$ .

[6]

- ii. In 2008 the population density of the village was 108 people per  $\text{km}^2$  and in 2013 it was 128 people per  $\text{km}^2$ . Determine how well the model fits these figures.

[2]

- 5.

- i. Express  $\frac{16 + 5x - 2x^2}{(x + 1)^2(x + 4)}$  in partial fractions.

ii. It is given that

$$\frac{dy}{dx} = \frac{(16 + 5x - 2x^2)y}{(x+1)^2(x+4)}$$

and that  $y = \frac{1}{256}$  when  $x = 0$ . Find the exact value of  $y$  when  $x = 2$ . Give your answer in the form  $Ae^n$ .

[7]

6. Helga invests £4000 in a savings account. After  $t$  days, her investment is worth £ $y$ . The rate of increase of  $y$  is  $ky$ , where  $k$  is a constant.

(a) Write down a differential equation in terms of  $t$ ,  $y$  and  $k$ . [1]

(b) Solve your differential equation to find the value of Helga's investment after  $t$  days. Give your answer in terms of  $k$  and  $t$ . [4]

It is given that  $k = \frac{1}{365} \ln\left(1 + \frac{r}{100}\right)$  where  $r\%$  is the rate of interest per annum. During the first year the rate of interest is 6% per annum.

(c) Find the value of Helga's investment after 90 days. [2]

After one year (365 days), the rate of interest drops to 5% per annum.

(d) Find the total time that it will take for Helga's investment to double in value. [5]

7. A new bird species is introduced into a region where it has previously been absent. Initially 20 birds are introduced. The rate of increase of the number  $N$  of birds after  $t$  years is modelled by

$$\frac{1}{1000} (10\,000 - N^2).$$

(a) Show that  $N = \frac{100(3e^{0.2t} - 2)}{(3e^{0.2t} + 2)}$ . [7]

(b) Hence explain what will happen to the number of birds over a long period of time, as predicted by the model. [2]

(c) State one limitation of the model. [1]

8. A tank is shaped as a cuboid. The base has dimensions 10 cm by 10 cm. Initially the tank is empty. Water flows into the tank at  $25 \text{ cm}^3$  per second. Water also leaks out of the tank at  $4h^2 \text{ cm}^3$  per second, where  $h$  cm is the depth of the water after  $t$  seconds. Find the time taken for the water to reach a depth of 2 cm. [9]

9. A scientist is attempting to model the number of insects,  $N$ , present in a colony at time  $t$  weeks. When  $t = 0$  there are 400 insects and when  $t = 1$  there are 440 insects.

(a) A scientist assumes that the rate of increase of the number of insects is inversely proportional to the number of insects present at time  $t$ .  
(i) Write down a differential equation to model this situation. [1]

(ii) Solve this differential equation to find  $N$  in terms of  $t$ . [4]

(b) In a revised model it is assumed that  $\frac{dN}{dt} = \frac{N^2}{3988e^{0.2t}}$ . Solve this differential equation to find  $N$  in terms of  $t$ . [6]

(c) Compare the long-term behaviour of the two models. [2]

10. The gradient of the curve  $y = f(x)$  is given by the differential equation

$$(2x - 1)^3 \frac{dy}{dx} + 4y^2 = 0$$

and the curve passes through the point (1, 1). By solving this differential equation show that

$$f(x) = \frac{ax^2 - ax + 1}{bx^2 - bx + 1},$$

where  $a$  and  $b$  are integers to be determined.

[9]

11.

$$\frac{dy}{dx} = \frac{x^2 \sin 2x}{2 \cos^2 4y - 1}.$$

The gradient function of a curve is given by

(a) Find an equation for the curve in the form  $f(y) = g(x)$

[6]

The curve passes through the point  $(\frac{1}{4}\pi, \frac{1}{12}\pi)$ .

(b) Find the smallest positive value of  $y$  for which  $x = 0$ .

[4]

12.

As a spherical snowball melts its volume decreases. The rate of decrease of the volume of the snowball at any given time is modelled as being proportional to its volume at that time. Initially the volume of the snowball is  $500 \text{ cm}^3$  and the rate of decrease of its volume is  $20 \text{ cm}^3$  per hour.

(a) Find the time that this model would predict for the snowball's volume to decrease to  $250 \text{ cm}^3$ .

[7]

(b) Write down one assumption made when using this model.

[1]

(c) Comment on how realistic this model would be in the long term.

[1]

END OF QUESTION paper

# Mark scheme

Question	Answer/Indicative content	Marks	Part marks and guidance	
1	<p>i <math display="block">\int \frac{1}{\theta + 20} d\theta = \int -k dt</math></p> <p>Separating variables</p> <p>ii <math>\ln(\theta + 20) = -kt (+ c)</math> or equivalent</p> <p>iii <math>\theta = Ae^{-kt} - 20</math> oe (i.e. <math>0 = e^{-kt+c} - 20</math>)</p>	<p>M1</p> <p>A1</p> <p>A1</p>	<p><math display="block">\frac{dt}{d\theta} = -\frac{1}{k(\theta + 20)}</math></p> <p>or invert each side:</p> <p>"Eqn A"</p> <p>"Eqn B"</p> <p><b>Examiner's Comments</b></p> <p>The majority of candidates used the method of 'separating the variables' and generally integrated each side correctly. However, many did not go further or failed to rearrange this equation correctly in order to express <math>\theta</math> in terms of <math>t</math>, <math>k</math> and an arbitrary constant. Many candidates thought that '+ c' should appear only at the end. A very few inverted each side and were generally less successful because of the position of 'k' immediately alongside the <math>\theta + 20</math> in the denominator.</p>	<p><math display="block">\frac{1}{\theta + 20}</math></p> <p>Must see <math>\theta + 20</math>; ignore posn 'k'</p>
	<p>ii <math>(-)3 = -k(40 + 20)</math></p> <p>iii <math>k = \frac{1}{20}</math> oe</p>	<p>M1</p> <p>*A1</p>	<p><math display="block">\frac{d\theta}{dt} = (-)3</math></p> <p>Using <math>t = 0</math>, <math>\theta = 40</math>, in given equation</p> <p><math display="block">k = -\frac{1}{20}</math></p> <p>Not</p> <p><b>Examiner's Comments</b></p> <p>There were some very good and neat solutions to this part, particularly by those who read the question carefully before delving into its solution. Even though <math>k</math> was defined to be a</p>	

				<p>positive constant, a large number of candidates obtained <math>k = -\frac{1}{20}</math>. A few interpreted the statement "...at this instant, the liquid is cooling...." to imply that there was a constant decrease of 3 degrees every minute.</p> <p>However, many candidates attempted to use all the information correctly and found <math>k</math>, their own constant of integration and, finally, the time taken for the liquid to freeze.</p>	
	ii	Subst $t = 0, \theta = 40$ & their $k$ (where necessary) into their Eqn A or their Eqn B and solve for the arbitrary constant	M1		
	ii	Subst $\theta = 0$ & their values of $k$ and the arbitrary constant into their Eqn A or their Eqn B	M1		
	ii	$t = 21.9722 = 22$ minutes cao www	dep* A1		
	iii	$k$ is larger	B1	<p><b>Examiner's Comments</b></p> <p>Most candidates stated that <math>k</math> must be larger, but explanations were sketchy.</p>	
		<b>Total</b>	<b>9</b>		
2	i	$\frac{dr}{dt} = \frac{k}{\sqrt{r}}$ oe	B2	$\frac{dr}{dt} = \dots$ ; B1 for $\frac{k}{\sqrt{r}}$	SR: B1 for $\frac{dr}{dt} \propto \frac{1}{\sqrt{r}}$
	i	Sep variables of their diff eqn (or invert) & integrate each side, increasing powers by 1 (or $\frac{1}{r} \rightarrow \ln r$ )	*M1	their d.e. must be $\frac{dr}{dt}$ (or $\frac{dt}{dr}$ ) = $f(r)$	Ignore absence of '+c' after integration

	<p>i</p> <p>Subst <math>\frac{dr}{dt} = 1.08, r = 9</math> into</p> <p>their diff eqn to find <math>k</math></p> <p>i</p> <p>Substitute <math>t = 5, r = 9</math> to find 'c'</p> <p>i</p> <p>Correct value of c (probably = 1.8 or -1.8)</p> <p>i</p> $r = (4.86t + 2.7)^{\frac{2}{3}}_{SW}$ <p>i</p>		<p>M1</p> <p>dep*M1</p> <p>A1</p> <p>A1</p> <p><b>Examiner's Comments</b></p> <p>The section in the specification topic "First Order Differential Equations" requires candidates to formulate a simple statement involving a rate of change as a differential equation. This process is by no means understood by the majority of candidates — and particularly not in this case where inverse proportion was involved. Many of the candidates thought that</p> $\frac{1}{k}$ <p>the constant should be inverse, i.e. <math>\frac{1}{k}</math>, which would have worked through satisfactorily had they not thought that the inverse aspect was now fulfilled and they could just write</p> $\frac{1}{k} \sqrt{r}$ <p>The separation of variables and subsequent integration were performed reasonably. At the end, many candidates failed to express <math>r</math> in terms of <math>t</math>, as directed.</p>	<p><math>\frac{dr}{dt}</math> (or <math>\frac{dt}{dr}</math>), <math>r</math> &amp; <math>k</math></p> <p>(✓ <math>k = 3.24</math> but M mark, not A)</p> <p>Must involve '+c' here</p> <p>Check other values</p> <p>Answer required in form <math>r = f(t)</math></p>	
	<p>ii</p> <p>ii</p>	<p>subst <math>t = 0</math> into any version of (i) result to find finite <math>r</math></p> <p>Any <math>V</math> in range <math>30.5 \leq V &lt; 30.55</math>, but not fortuitously</p>	<p>M1</p> <p>A1</p>	<p>Accept <math>9.72\pi</math> or <math>\frac{243}{25}\pi</math></p>	<p>(✓ <math>r \approx 1.938991\dots</math> but M mark, not A)</p>



			Examiner's Comments
			Most understood what to do, gaining the method mark, and a few obtained the correct result.
Total			9
3	<p>i <math>\frac{dV}{dt} = \pm 0.01</math></p> <p>i by similar triangles, <math>\frac{h}{4.5} = \frac{r}{3}</math></p> <p>i <math>\frac{dh}{dt} = \pm 0.01 \times \text{their } \frac{dh}{dV} \text{ oe}</math></p> <p>i <math>-0.01 = \left(\frac{4}{9} \pi h^2\right) \times \frac{dh}{dt} \text{ oe soi}</math></p>	<p>B1</p> <p>B1 <math>r = \frac{2h}{3} \text{ oe}</math> may be implied by</p> <p>B1 <math>\frac{dV}{dh} = \frac{4}{9} \pi h^2 \text{ oe}</math></p> <p>M1 use of Chain rule</p> <p>completion to given result www</p> <p><u>Examiner's Comments</u></p> <p>This defeated all but the best candidates. Many scored an</p> <p>easy B1 for <math>\frac{dV}{dt} = -0.01</math> (although a</p> <p>good number missed out on this because they wrote</p> <p>A1 <math>\frac{dh}{dt} = -0.01</math> (instead).</p> <p>Thereafter very few made any progress: the need to eliminate <math>r</math> was often not appreciated. Those who did spot the relationship between <math>h</math> and <math>r</math> often went on to score full marks. Having obtained <math>V</math> in terms of <math>h</math> only, a few strong</p>	<p>may follow from incorrect differentiation: expressions must be a function of either <math>r</math> or <math>h</math> or both</p> <p><math>h^2 \frac{dh}{dt} = \frac{-0.09}{4\pi} = \frac{-9}{400\pi}</math></p>

				<p>candidates derived the required result by considering <math>V(t) = 13.5\pi - 0.01t</math> and differentiating.</p>	
ii	$\int h^2 dh = \int \frac{-9}{400\pi} dt$	M1	separation of variables		if no subsequent work, integral signs needed, but allow omission of $dh$ or $dt$ , but must be correctly placed if present;
ii	$\frac{h^3}{3} = \frac{-9}{400\pi} t(+c)$	A1			
ii	substitution of $t = 0$ and $h = 4.5$ in their expression following integration	M1	expression must include $c$ and powers must be correct on each side		
			allow $-0.0215$ or $-0.02148591\dots$ r.o.t to 4 sf or more and similarly $91.125$		
ii	$h = \sqrt[3]{\frac{729}{8} - \frac{27t}{400\pi}}$ oe isw	A1	<p><u>Examiner's Comments</u></p> <p>This proved surprisingly difficult for many. Those who did separate the variables either differentiated <math>h^2</math> instead of integrating, or omitted the constant of integration and made no further progress. Those who did achieve a correct value for "<math>c</math>" often went on to spoil their answer, or simply left the equation in implicit form.</p>		$91.125 = \frac{729}{8}$
iii	set $h = 0$ and solve to obtain positive $t$	M1	$\frac{1}{3} \pi \times 3^2 \times 4.5 \div 0.01$ $(= 1350\pi)$		NB $1350\pi = 4241.150082\dots$
iii	71 minutes cao	A1	<p><u>Examiner's Comments</u></p> <p>Most realised that setting <math>h = 0</math> was required here. Due to earlier errors, this sometimes led to a negative value for <math>t</math>; surprisingly this did not always set the alarm bells ringing. A good proportion of candidates who did everything right lost an easy mark because they failed to convert the answer to</p>		

				minutes. Some candidates who had made no progress earlier, were astute enough to realise that the correct answer could be obtained without the result from part (ii), as the rate of change of volume was constant.	
		Total	11		
4	i	$\frac{dP}{dt} = \frac{k}{P}$	B1	$\frac{dP}{dt} = \frac{1}{kP}$	$k$ should be unspecified at this stage
	i	$k = 100 \text{ from } \frac{dP}{dt} = \frac{k}{P}$	B1	$\text{or } k = 0.01 \text{ from } \frac{dP}{dt} = \frac{1}{kP}$	may be seen later
	i	$\int P dP = \int (\text{their } k) dt$	M1*	allow $k = 1$	allow omission of $\int$ and recovery of omission of one operator for <b>M1*A1</b>
	i	$\frac{P^2}{2} = kt + c$	A1	$t = \frac{P^2}{2k} + d$	if <b>M0, SC2</b> for $\ln P = kt + c$ thereafter only <b>M1</b> may be earned
	i	substitution of $t = 0$ and $P = 100$	M1 dep*	may follow incorrect algebraic manipulation, but equation must include $c$ (or $d$ )	NB $c = 5000$ or $d = -50$
	i	$P = \sqrt{10000 + 200t} \text{ or } 10\sqrt{100 + 2t}$	A1		allow recovery from eg use of $x$ for $P$ throughout, but withhold final <b>A1</b> for
	i	$\text{or } P = \sqrt{200(50 + t)} \text{ isw cao}$			<b>eg</b> $x = \sqrt{10000 + 200t}$
				<u>Examiner's Comments</u>	
				A surprising number of candidates did not seem to understand inverse proportion, and setting up the initial equation elicited a wide range of incorrect responses. Those who did set up the equation correctly usually went on to earn at least four marks out of six. Finding $k$ caused more difficulty than expected: many candidates mistakenly assuming that $t = 1, P = 101$ was a valid pair of values, instead of working with the information given in the question. In some cases	

				rearranging to make $P$ the subject of the formula proved troublesome.	
	ii	$t = 8, P = 107.7$ or $108$ so model was a good fit in 2008 oe	B1	or $t = 8.3(2)$ when $P = 108$ + comment	value of $P$ or $t$ must be found and correct comment made in each case; comments may be in same sentence.
	ii	$t = 13, P = 112(.2)$ , so model was not appropriate in 2013 oe	B1	or $t = 31.9(2)$ or $32$ when $P = 128$ + comment  comments may be in same sentence, but both values must be referenced	if <b>B0B0, SC1</b> for both values found no FT marks available  comments on trends, extrapolation etc do not score just ticks / crosses etc do not score
	ii			<u>Examiner's Comments</u>  Full marks were rarely achieved in this case. Those who did find the correct values often speculated on future trends rather than commenting on the two values they had found. It was necessary to have earned at least four marks in part (i) to score in this part.	
		<b>Total</b>	<b>8</b>		
5	i	$\frac{A}{(x+4)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$	B1		may be awarded later
	i	$[16 + 5x - 2x^2] = A(x+1)^2 + B(x+1)(x+4) + C(x+4)$	M1		allow sign errors only
	i	$A = -4$	A1	<b>NB</b> $36 = -9A$	if <b>B0M0</b> , allow <b>SC3</b> for
	i	$C = 3$	A1	$9 = 3C$	$\frac{2x+5}{(x+1)^2} - \frac{4}{x+4}$

	i	$B = 2$ isw	A1	$-2 = A + B, 5 = 2A + 5B + C, 16 = A + 4B + 4C$ <b>NB</b> $\frac{-4}{(x+4)} + \frac{2}{(x+1)} + \frac{3}{(x+1)^2}$	<b>Examiner's Comments</b> Most recognised the correct form of partial fractions and successfully cleared the fractions to produce a fully correct solution.
	ii	$\int \frac{dy}{y} = \int \frac{16 + 5x - 2x^2}{(x+1)^2(x+4)} dx$ $\frac{3}{(x+1)^2} + \frac{2}{(x+1)} - \frac{4}{(x+4)}$ <p>seen in RHS, may be embedded</p> $\frac{-3}{x+1} + 2\ln(x+1) - 4\ln(x+4) + c$ $\ln\left(\frac{1}{256}\right) = -3 + 2\ln 1 - 4\ln 4 + c$ <p><math>c = 3</math> cao</p> $\ln y = \frac{-3}{2+1} + 2\ln(2+1) - 4\ln(2+4) + 3$	B1 M1* A1FT M1*dep A1 M1*dep	separation of variables <b>FT</b> their partial fractions if two or three terms; ignore LHS <b>FT</b> their non-zero 3, 2 and 4; allow recovery from $x + 1^2$ in denominator; if brackets in log terms omitted, allow <b>A1</b> if recovery seen in substitution substitution of $x = 0$ and $y = \frac{1}{256}$ allow if error in manipulation following integration; <b>or</b> $A = e^{-3}$ from $y = Ae^{\frac{-3}{x+1}} \frac{(x+1)^2}{(x+4)^4}$ substitution of $x = 2$ ; dependent on award of previous <b>M1M1</b> and numerical value found	allow omission of integral signs; allow omission of $dy$ or $dx$ but not both may be implied by correct integration of two of their terms allow omission of $+ c$ here $+ c$ must be included and LHS must be correctly obtained allow <b>M1</b> if substitution follows incorrect manipulation

	ii	$y = \frac{e^2}{144}$ oe	A1	for $c$	eg to find expression for $y$  <b>Examiner's Comments</b>  The separation of variables caused problems for many, but most recognised the link with part (i) and worked with their partial fractions. The method was often well understood, but only a minority of candidates had the stamina and attention to detail to go on to the end and achieve the correct result.
		Total	12		
6	a	$\frac{dy}{dt} = ky$	B1(AO3.1b)		
	b	$\frac{dy}{y} = kdt$ $[\ln y]_{4000}^y = k[t]_0^t$ or $\ln y = kt + c$ $\ln \frac{y}{4000} = kt$ or $\ln 4000 = 0 + c$ $y = 4000e^{kt}$	M1(AO1.1a) M1(AO1.1) A1(AO1.1) A1(AO1.1) [4]	Attempt separation of variables Correct integrals and limits Correct substitution in correct integral	
	c	$4000e^{\frac{90}{365} \ln 1.06}$ $= 4057.89$	M1(AO1.1) A1(AO1.1) [2]	FT their part (b) BC	

	d	<p>After 1 year, increased by factor 1.06</p> $\frac{2}{1.06}$ <p>Require further increase by factor</p> $e^{\frac{t}{365} \ln 1.05} = \frac{2}{1.06}$ $\frac{t}{365} \ln 1.05 = \ln \frac{2}{1.06}$ $t = \frac{365}{\ln 1.05} \times \ln \frac{2}{1.06}$ $= 4750$ <p>Total number of days = 5115</p>	<p>M1(AO3.1b)</p> <p>M1(AO1.1)</p> <p>A1(AO2.1)</p> <p>M1(AO1.1)</p> <p>A1(AO3.2a)</p> <p>[5]</p>	<p>May be implied</p> <p>Attempt to form equation with 1.05 and 1.06</p> <p>Correct equation</p> <p>Attempt to remove logs</p> <p>isw</p>	<p>OR</p> <p>BC</p>	
Total			12			
7	a	$\frac{dN}{dt} = \frac{1}{1000} (10000 - N^2)$ $\frac{dN}{10000 - N^2} = \frac{1}{1000} dt$ $\frac{1}{200} \left( \frac{1}{100 - N} + \frac{1}{100 + N} \right) dN = \frac{1}{1000} dt$ $\ln \frac{100 + N}{100 - N} = 0.2t + c$	<p>M1(AO 3.3)</p> <p>M1(AO 1.1)</p> <p>M1(AO 3.1b)</p> <p>A1(AO 2.1)</p> <p>M1(AO 1.1)</p>	<p>Attempt separate variables</p> <p>Attempt PFs with correct denominators</p> <p>Equation of form</p>		

		<p>when <math>t = 0</math>, <math>N = 20</math> hence <math>c = \ln(3/2)</math></p> $\ln \frac{2(100+N)}{3(100-N)} = 0.2t$ $\frac{2(100+N)}{3(100-N)} = e^{0.2t}$ $N = \frac{100(3e^{0.2t} - 2)}{(3e^{0.2t} + 2)} \quad \mathbf{AG}$	<p>A1(AO 2.1)</p> <p>A1(AO 1.1)</p> <p>[7]</p>	$k \ln \frac{100+N}{100-N} = k't (+ c)$ <p>Allow without <math>+ c</math></p> <p>Attempt find <math>c</math> with <math>t = 0</math> and <math>N = 20</math></p> <p>Any correct equation in <math>N</math> &amp; <math>t</math> after antilog</p> <p>Correctly rearrange to give answer</p>			
	b	No. of birds increases and tends to 100	<p>E1(AO 2.2b)</p> <p>E1(AO 3.4)</p> <p>[2]</p>	<table border="1"> <tr> <td style="width: 20px; height: 40px;"></td> <td style="width: 20px; height: 40px;"></td> </tr> </table>			
	c	Eg $N$ is discrete but modelled by continuous	<p>E1(AO 3.5b)</p> <p>[1]</p>	<table border="1"> <tr> <td style="width: 100px; height: 40px;">No account of external factors eg weather</td> <td style="width: 100px; height: 40px;"></td> </tr> </table>	No account of external factors eg weather		
No account of external factors eg weather							
		Total	10				
8		$V = 100h \Rightarrow \frac{dv}{dh} = 100$ $\frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt} = 100 \frac{dh}{dt} \quad [= 25 - 4h^2]$	<p>M1(AO 3.4)</p> <p>A1(AO 1.2)</p> <p>M1(AO 3.1b)</p>	<table border="1"> <tr> <td style="width: 100px; height: 60px;"></td> <td style="width: 100px; height: 60px;"></td> </tr> </table>			



$$\Rightarrow 25 - 4h^2 = 100 \frac{dh}{dt} \text{ oe}$$

$$\Rightarrow \int_0^2 \frac{1}{25-4h^2} dh = \int_0^t \frac{1}{100} dt$$

$$\Rightarrow \frac{1}{10} \int_0^2 \frac{1}{5+2h} + \frac{1}{5-2h} dh = \int_0^t \frac{1}{100} dt$$

$$\Rightarrow \frac{1}{10} \times \frac{1}{2} [\ln(5+2h) - \ln(5-2h)]_0^2 = \frac{t}{100}$$

$$\Rightarrow 5 \ln 9 = t \text{ oe}$$

Time when depth is 2 cm is 11.0 seconds (3 sf)

M1(AO 2.5)

M1(AO 3.4)

A1(AO 2.1)

M1(AO 1.2)

A1(AO  
2.2a)

A1(AO  
3.2a)

[9]

Equate  $25 - 4t^2$   
to their  
 $\frac{dv}{dh} \times \frac{dh}{dt}$

Attempt  
integration with  
correct  
denominator on  
LHS

Attempt partial  
fractions with  
correct  
denominators  
on LHS

Correct partial  
fractions

Correct integral;  
ignore limits

Any correct  
numerical  
expression for  $t$

Allow 11  
seconds

10.9861...

Total		9	
9	a	<p>(i) <math>\frac{dN}{dt} = \frac{k}{N}</math></p>	<p>B1 (AO 3.3)</p> <p>State correct differential equation</p> <div style="border: 1px solid black; padding: 5px; display: inline-block;">       Or <math>\frac{dN}{dt} = \frac{1}{k'N}</math> or     </div> <p>equiv with <math>k</math> on LHS</p>
	a	<p>(ii)</p> <p><math>kt = \int NdN</math></p> <p><math>kt = \frac{1}{2}N^2 + c</math></p> <p><math>0 = 80000 + c \Rightarrow c = -80000</math></p> <p><math>k = 96800 - 80000 = 16800</math></p>	<p>[1]</p> <p><u>Examiner's Comments</u></p> <p>The majority of candidates could state a correct differential equation, including a coefficient of proportionality. The most common error was to have the equation as a function of <math>t</math> not of <math>N</math>.</p> <p>M1* (AO 2.1)</p> <p>Attempt integration</p> <p>Obtain equation of form <math>at = bN^2 + c</math> Condone no <math>+ c</math></p> <p>M1d* (AO 3.4)</p> <p>Attempt <math>c</math> from (0, 400)</p> <p>Substitute (0, 400) into their equation containing <math>c</math> and <math>k</math> Could give value for <math>c</math>, or could result in an equation involving both <math>c</math> and <math>k</math></p> <p>M1d* (AO 3.4)</p>

$$N = \sqrt{(33600t + 160000)}$$

A1  
(AO 1.1)

[4]

Attempt  $k$  from  
(1, 440)

depending on  
structure

Substitute (1,  
440) into their  
equation  
containing  $c$   
(possibly now  
numerical) and  $k$   
If  $c$  is numerical  
then value of  $k$   
must be  
attempted  
If this gives  
second  
equation in  $c$   
and  $k$  then the  
equations need  
to be solved  
simultaneously  
for  $c$  and  $k$  to  
award M1

Correct  
equation for  $N$

$N$  must be the  
subject of the  
equation

Examiner's Comments

Candidates who had stated a correct equation in part (a) were usually able to rearrange and integrate correctly. They were then able to use the given boundary conditions to find a particular solution to the differential equation. Having got this

far, a few candidates did not gain the final mark due to making errors when rearranging the equation to the requested form, such as square rooting term by term.

$$\int 3988N^2 dN = \int e^{-0.2t} dt$$

$$-3988N^{-1} = -5e^{-0.2t} + c$$

OR

$$-N^{-1} = -\frac{5}{3988}e^{-0.2t} + c$$

$$-9.97 = -5 + c \Rightarrow c = -4.97$$

OR

$$-\frac{1}{400} = -\frac{5}{3988} + c \Rightarrow c = -\frac{497}{398800}$$

**M1**  
(AO 3.1a)

Separate variables and attempt integration

Must be valid method to separate variables so allow coefficient slips only  
Some attempt to integrate, but may not be correct  
BOD if no integral signs, as long as integration is actually attempted

**M1\***  
(AO 1.1a)

Integrate to obtain answer of correct form

Obtain integral of the form  $aN^{-1} = be^{-0.2t} + c$  or equiv

**A1**  
(AO 1.1)

Obtain correct integral

Condone no + c  
Any equivalent form

**M1d\***  
(AO 2.2a)

Attempt c from (0, 400) or (1, 440)

As far as attempting numerical value for c

$$\frac{3988}{N} = 5e^{-0.2t} + 4.97$$

OR

$$\frac{1}{N} = \frac{5}{3988} e^{-0.2t} + \frac{497}{398800}$$

$$N = \frac{3988}{5e^{-0.2t} + 4.97}$$

M1d\*  
(AO 1.1)

A1  
(AO 1.1)

[6]

Attempt to  
make  $N$  the  
subject

Correct  
equation for  $N$

NB (0, 400)  
gives  $-4.97$ , (1,  
440) gives an  
answer which  
rounds to  $-4.97$   
Equation may  
no longer be  
correct

Using correct  
algebraic  
processes  
throughout, but  
allow sign slips  
– this includes  
any  
rearrangement  
attempt made  
prior to  
attempting  $c$   
Must involve a  
 $c$ , either  
numerical or still  
as  $c$

Any correct  
equation of form  
 $N = \dots$

				<p><u>Examiner's Comments</u></p> <p>Candidates identified the need to separate the variables before integrating and many made a good attempt to do so. The majority of candidates chose to leave the 3988 where it was rather than incorporating it with the term in <math>N</math>. This caused problems for some when rewriting the right-hand side in a form that could be integrated, with <math>3988e^{-0.2t}</math> being the most common error. When finding a value for the constant of integration most candidates opted to use the first condition given. Once again, not all candidates were able to use the correct method to find <math>N</math> in terms of <math>t</math>; the most common error was to simply invert each of the three terms in their equation.</p>					
	c	<p>Model in <b>(a)</b> predicts that population will continue to increase</p> <p>Model in <b>(b)</b> predicts that population will tend towards a limit of 802</p>	<p>E1 (AO 3.5a)</p> <p>E1 (AO 3.5a)</p>	<table border="1"> <tr> <td data-bbox="1128 608 1393 1182"> <p>Comment about continuing to increase</p> </td> <td data-bbox="1393 608 1664 1182"> <p>Allow comments such as tending to infinity Any additional comments must also be correct so E0 for eg 'will always increase at a steady rate' but E1 for 'will always increase but the rate of increase will decrease'</p> </td> </tr> <tr> <td data-bbox="1128 1182 1393 1431"> <p>Comment about tending towards a limit of 802</p> </td> <td data-bbox="1393 1182 1664 1431"> <p>Allow a limit of 803 or 802.4 Must come from a fully correct function</p> </td> </tr> </table>	<p>Comment about continuing to increase</p>	<p>Allow comments such as tending to infinity Any additional comments must also be correct so E0 for eg 'will always increase at a steady rate' but E1 for 'will always increase but the rate of increase will decrease'</p>	<p>Comment about tending towards a limit of 802</p>	<p>Allow a limit of 803 or 802.4 Must come from a fully correct function</p>	
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<p>Comment about tending towards a limit of 802</p>	<p>Allow a limit of 803 or 802.4 Must come from a fully correct function</p>								

			[2]	<table border="1"> <tr> <td></td> <td>Any additional comments must also be correct</td> </tr> </table> <p><u>Examiner's Comments</u></p> <p>When considering the first model, candidates simply needed to identify that it predicted that the population would continue to increase and a number of candidates were able to do so. Some spoilt an otherwise correct statement by including additional, incorrect detail such as increasing at a faster rate. Only the most able candidates were able to both identify that the second model would predict that the population will tend towards a limit and also identify the value of the limit. There was no credit for commenting on how realistic each of the two models was, but some candidates did consider this.</p>		Any additional comments must also be correct			
	Any additional comments must also be correct								
		Total	13						
10		$(2x-1)^3 \frac{dy}{dx} + 4y^2 = 0$ $-\frac{1}{4} \int \frac{dy}{y^2} = \int \frac{dx}{(2x-1)^3}$ $\int \frac{dy}{y^2} = -\frac{1}{y}$ $\int \frac{dx}{(2x-1)^3} = \frac{(2x-1)^{-2}}{(2)(-2)}$	M1(AO 2.5)E  A1(AO 1.1)E  M1(AO 1.1)E A1(AO 1.1)C	<table border="1"> <tr> <td>Attempt to separate variables</td> <td></td> </tr> <tr> <td>M1 for <math>k(2x-1)^{-2}</math></td> <td></td> </tr> </table>	Attempt to separate variables		M1 for $k(2x-1)^{-2}$		
Attempt to separate variables									
M1 for $k(2x-1)^{-2}$									

	$\frac{1}{4y} = -\frac{1}{4(2x-1)^2} + c, (1,1) \Rightarrow c = \dots$ $\frac{1}{y} = -\frac{1}{(2x-1)^2} + 2$ $\frac{1}{y} = \frac{2(2x-1)^2 - 1}{(2x-1)^2}$ $y = \frac{(2x-1)^2}{2(2x-1)^2 - 1}$ $y = \frac{4x^2 - 4x + 1}{8x^2 - 8x + 1}$	<p>M1(AO 2.1)C</p> <p>A1(AO 2.2a)A</p> <p>M1(AO 3.1a)A</p> <p>M1(AO 1.1)A</p> <p>A1(AO 2.2a)A</p> <p>[9]</p>	<p>Use of (1, 1) to find <math>c</math> – dependent on the previous two M marks and substituted into correct form</p> <p>Oe</p> <p>Or re-write in terms of <math>y</math></p> <p>Correct method for combining both terms on rhs (dependent on previous M mark) before taking the reciprocal</p> <p>Taking the reciprocal (dependent on previous M marks) and making <math>y</math> the subject  <math>a = 4, b = 8</math></p> <p>Remove tripledecker fractions</p>	
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				<p><b>Examiner's Comments</b></p> <p>The responses to this final question in the pure section were mixed with examiners reporting a mixture of excellent responses followed by those that struggled with both the integration and the resulting algebraic manipulation required to obtain the answer in the required form. While most correctly separated the variables and wrote</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td style="text-align: center;"> <math display="block">-\frac{1}{4} \int \frac{dy}{y^2} = \int \frac{dx}{(2x-1)^3}</math> </td> <td style="padding-left: 10px; vertical-align: middle;">many candidates</td> </tr> </table> <p>had issues with the placement of the fraction on the left-hand side with examiners reporting that frequently this became a 4 rather than remaining as a quarter. While many candidates correctly integrated and remembered to include an arbitrary constant many decided to re-arrange their equation before attempting to find this constant; candidates are advised that in the majority of situations it is probably wisest to work out the +c immediately. Of those that obtained a correct particular solution to this differential equation, for example,</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td style="text-align: center;"> <math display="block">\frac{1}{y} = -\frac{1}{(2x-1)^2} + 2</math> </td> <td style="padding-left: 10px; vertical-align: middle;">many did not know</td> </tr> </table> <p>the correct method for obtaining the result for f(x) in the required form. Many candidates took the reciprocal of each term separately rather than combining all relevant fractions first before taking the reciprocal and then expanding the brackets.</p>	$-\frac{1}{4} \int \frac{dy}{y^2} = \int \frac{dx}{(2x-1)^3}$	many candidates	$\frac{1}{y} = -\frac{1}{(2x-1)^2} + 2$	many did not know	
$-\frac{1}{4} \int \frac{dy}{y^2} = \int \frac{dx}{(2x-1)^3}$	many candidates								
$\frac{1}{y} = -\frac{1}{(2x-1)^2} + 2$	many did not know								
		<b>Total</b>	<b>9</b>						
11	a	$\int (2 \cos^2 4y - 1) dy = \int x^2 \sin 2x dx$  $2 \cos^2 4y - 1 = \cos 8y$	M1 (AO 1.1a)  M1 (AO	<table border="1" style="width: 100%;"> <tr> <td style="width: 50%; text-align: center; vertical-align: middle;">Separate variables</td> <td style="width: 50%;"></td> </tr> </table>	Separate variables				
Separate variables									

		$\int \cos 8y dy = \frac{1}{8} \sin 8y + c_1$ $\int x^2 \sin 2x dx = -\frac{1}{2} x^2 \cos 2x + \int x \cos 2x dx$ $= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x dx$ $= -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + c_2$ $\frac{1}{8} \sin 8y = -\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + c$	<p>3.1a)</p> <p>A1 (AO 1.1)</p> <p>M1 (AO 3.1a)</p> <p>M1 (AO 1.1)</p> <p>A1 (AO 1.1)</p> <p>6]</p>	<p>Attempt use of double angle formula</p> <p>Obtain correct integral</p> <p>Attempt integration by parts once</p> <p>Attempt second integration by parts</p> <p>Obtain correct integral</p>	<p>Obtain <math>\pm \cos 8y</math></p> <p>Condone no + <math>c_1</math></p> <p>Condone no + <math>c_2</math></p>	
	b	$\sin \frac{8}{12} \pi = 4 \left( \frac{1}{4} \pi \right)^2 \cos \frac{\pi}{2} + 4 \left( \frac{1}{4} \pi \right) \sin \frac{\pi}{2} + 2 \cos \frac{\pi}{2} + c.$ $c = \frac{1}{2} \sqrt{3} - \pi$	<p>M1 (AO 1.1)</p> <p>A1 (AO 1.1)</p>	<p>Attempt <math>c</math>, using</p> $x = \frac{1}{4} \pi,$ $y = \frac{1}{12} \pi$ <p>Obtain</p>		



$$\int -0.04 dt = \int \frac{1}{V} dV$$

$$-0.04t = \ln V + c$$

$$c = -\ln 500$$

$$-0.04t = \ln 250 - \ln 500$$

$$t = 17.3 \text{ hours}$$

Alternate method

$$\frac{dV}{dt} = -kV$$

$$-20 = -k \times 500 \text{ so } k = 0.04$$

$$\int_0^T -0.04 dt = \int_{500}^{250} \frac{1}{V} dV$$

A1(AO 1.1)

Separate variables and attempt integration

M1\* (AO 3.4)

Correct integral – could still be in terms of  $k$

Accept no +  $c$  here

M1dep\*(AO 3.4)

A1 (AO 3.3)

Use  $t = 0$ ,  $V = 500$  to find  $c$

[7]

Attempt to find  $t$  when  $V = 250$

B1(AO 3.3)

Obtain 17.3 hours, or better (17 hours and 20 minutes)

Units needed 17.3286...

B1(AO 3.3)

M1(AO 1.1a)

M1(AO 3.4)

Set up correct differential equation

Allow  $k$  for  $-k$

M1(AO 3.4)

Correct value for  $k$  – may be seen later

Or  $k = -0.04$

A1(AO 1.1)

A1(AO 3.4)

[7]

Separate variables and attempt integration of LHS

OR

Use of  $t = 0$ ,  $V = 500$

Use of  $t$  limits 0 and  $T$  (accept  $t =$

		$-0.04T = -0.693\dots$ $T = 17$ hours		<table border="1"> <tr> <td>Use of <math>t = T</math>, <math>V = 250</math> (accept <math>t = t</math>)</td> <td><math>t</math></td> </tr> <tr> <td>Obtain 17.3 hours, or better (17 hours and 20 minutes)</td> <td>Use of <math>V</math> limits 500 and 250 (either way round)  Units needed 17.3286...</td> </tr> </table>	Use of $t = T$ , $V = 250$ (accept $t = t$ )	$t$	Obtain 17.3 hours, or better (17 hours and 20 minutes)	Use of $V$ limits 500 and 250 (either way round)  Units needed 17.3286...	
Use of $t = T$ , $V = 250$ (accept $t = t$ )	$t$								
Obtain 17.3 hours, or better (17 hours and 20 minutes)	Use of $V$ limits 500 and 250 (either way round)  Units needed 17.3286...								
	b	E.g. Assumes that temperature remains constant E.g. Assume that the snowball remains a sphere throughout	<b>B1(AO 3.5b)</b>  <b>[1]</b>	<table border="1"> <tr> <td>Any valid assumption made</td> <td></td> </tr> </table>	Any valid assumption made				
Any valid assumption made									
	c	Not very realistic as volume never equals 0, so snowball never melts completely	<b>B1(AO 3.5b)</b>  <b>[1]</b>	<table border="1"> <tr> <td>Consider long term prediction</td> <td></td> </tr> </table>	Consider long term prediction				
Consider long term prediction									
		<b>Total</b>	<b>9</b>						