1. Fig. 9 shows the line $y = x$ and the curve $y = f(x)$, where $f(x) = \frac{1}{2}(e^x - 1)$. The line and the curve intersect at the origin and at the point P(a, a).

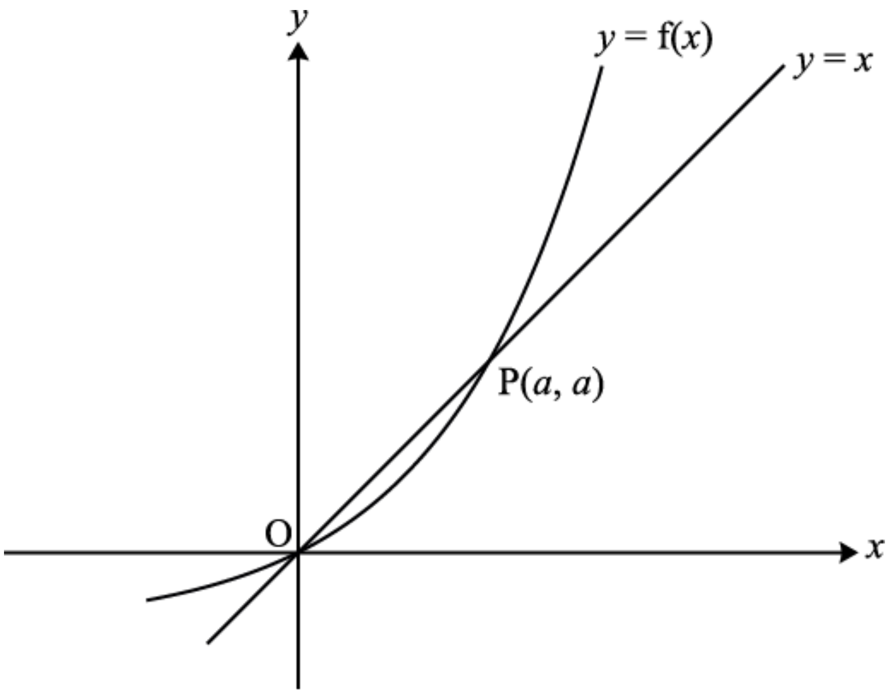

Fig. 9

i. Show that $e^a = 1 + 2a$.

[1]

ii. Show that the area of the region enclosed by the curve, the x-axis and the line $x = a$ is $\frac{1}{2}a$. Hence find, in terms of a, the area enclosed by the curve and the line $y = x$.

[6]

iii. Show that the inverse function of f(x) is g(x), where $g(x) = \ln(1 + 2x)$. Add a sketch of $y = g(x)$ to the copy of Fig. 9.

[5]

iv. Find the derivatives of f(x) and g(x). Hence verify that $g'(a) = \frac{1}{f'(a)}$.

Give a geometrical interpretation of this result.

[7]

2. Evaluate $\int_0^{\frac{\pi}{12}} \cos 3x \, dx$, giving your answer in exact form. [3]

3. **In this question you must show detailed reasoning.**
The shaded region in Fig. 2 is bounded by the curves $y = \sin 3x$ and $y = 3 \sin 3x$.

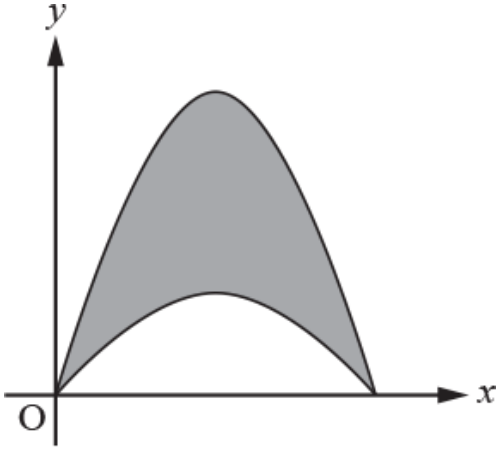

Fig. 2

Show that the area of this region is $\frac{4}{3}$. [4]

4. Find $\int 12e^{3x} dx$. [2]

5. **In this question you must show detailed reasoning.**

Fig. 6 shows the curves $y = \sin x$ and $y = \cos x$ for $-\pi \le x \le \pi$.

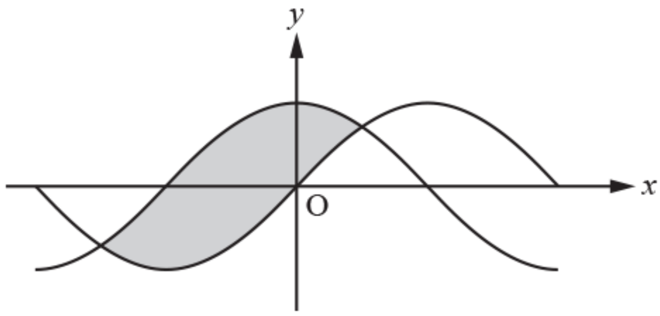

Fig. 6

[4]

Show that the area of the region enclosed between the curves is given by
(a) $\int_a^b (\cos x - \sin x) \, dx$, where the values of a and b are to be determined.

(b) Find the exact value of the area of this region. [3]

6. (a) Show that $8\sin^2 x\cos^2 x$ can be written as $1 - \cos 4x$. [3]

(b) Hence find $\int \sin^2 x\cos^2 x\,dx$. [3]

7. **In this question you must show detailed reasoning.**

Fig. 15 shows the graph of $y = 5xe^{-2x}$. Find the exact value of the area of the shaded region between the curve, the x-axis and the line parallel to the y-axis through the maximum point on the curve. [9]

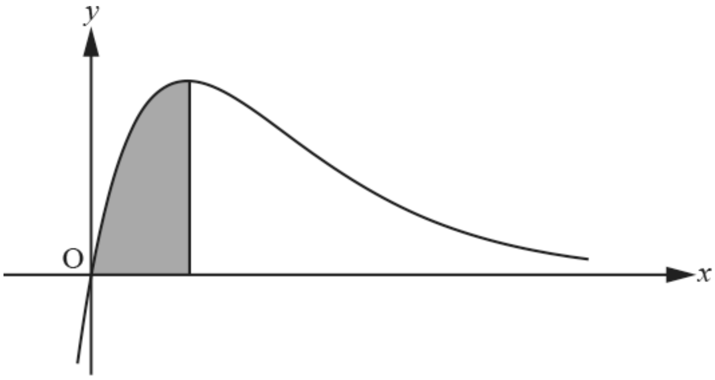

Fig. 15

8. **In this question you must show detailed reasoning.**

Fig. 9 shows the line $y = x$ and the curve $y = \frac{4}{3}x^3 + a$ where a is a constant. The line is a tangent to the curve, touching the curve in the first quadrant.

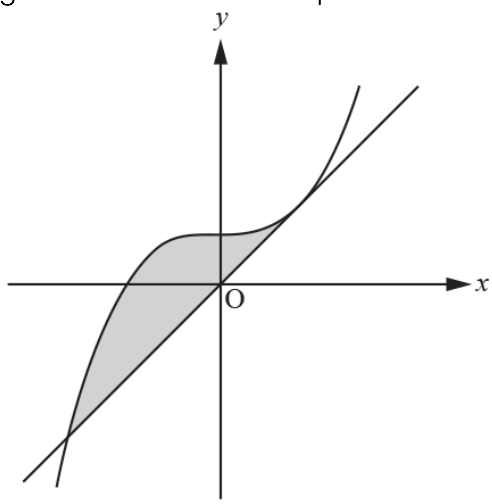

Fig.9

(a) (i) Show that, at the point of contact between the line and the curve, $4x^2 = 1$. [2]

(ii) Hence find the value of a. [4]

(b) Find the area of the shaded region between the curve and the line. [7]

END OF QUESTION paper

Question			Answer/Indicative content	Marks	Part marks and guidance	
1		i	At P(a, a) g(a) = a so ½(e^a −1) = a		NB AG	
		i	⇒ $e^a = 1 + 2a$*	B1	**Examiner's Comments** This mark was usually earned.	
		ii	$A = \int_0^a \frac{1}{2}(e^x - 1)\,dx$	M1	correct integral and limits	limits can be implied from subsequent work
		ii	$= \frac{1}{2}\left[e^x - x\right]_0^a$	B1	integral of $e^x - 1$ is $e^x - x$	
		ii	= ½ (e^a − a e^0)	A1		
		ii	= ½ (1 + 2a − a −1) = ½ a*	A1	NB AG	
		ii	area of triangle = ½ a^2	B1		
		ii	area between curve and line = ½ a^2 − ½ a	B1cao	mark final answer **Examiner's Comments** Virtually everyone scored M1 for writing down the correct integral and limits, but many candidates made a meal of trying to integrate ½ (e^x − 1) , with ¼ (e^x − 1)2 not an uncommon wrong answer. Having successfully negotiated this hurdle, using part (i) to derive ½ a was spotted by about 50% of the candidates. Quite a few candidates managed to recover to earn the final 2 marks for ½ (a^2 − a) (without incorrectly simplifying this to½ a!).	

		iii	$y = ½(e^x − 1)$ swap x and y $x = ½ (e^y − 1)$			
		iii	⇒ $2x = e^y − 1$	M1	Attempt to invert — one valid step	merely swapping x and y is not 'one step'
		iii	⇒ $2x + 1 = e^y$	A1		
		iii	⇒ $\ln(2x + 1) = y$* ⇒ $g(x) = \ln(2x + 1)$	A1	$y = \ln(2x + 1)$ or $g(x) = \ln(2x + 1)$ **AG**	apply a similar scheme if they start with $g(x)$ and invert to get $f(x)$. or $g\,f(x) = g((e^x − 1)/2)$ M1
		iii	Sketch: recognisable attempt to reflect in			
		iii	$y = x$	M1	through O and (a, a)	$= \ln(1 + e^x − 1) = \ln(e^x)$ A1 $= x$ A1
		iii	Good shape	A1	no obvious inflexion or TP, extends to third quadrant, without gradient becoming too negative **Examiner's Comments** Finding the inverse function proved to be an easy 3 marks for most candidates – candidates are clearly well practiced in this. The graphs were usually recognisable reflections in y = x, but only well drawn examples – without unnecessary maxima or inflections – were awarded the 'A' mark.	similar scheme for fg See appendix for examples
		iv	$f'(x) = ½\, e^x$	B1		
		iv	$g'(x) = 2/(2x + 1)$	M1	$1/(2x + 1)$ (or $1/u$ with $u = 2x + 1$) …	
		iv		A1	… × 2 to get $2/(2x + 1)$	
		iv	$g'(a) = 2/(2a + 1)$, $f'(a) = ½\, e^a$	B1	either $g'(a)$ or $f'(a)$ correct soi	
		iv	so $g'(a) = 2/e^a$ or $f'(a) = ½\,(2a + 1)$	M1	substituting $e^a = 1 + 2a$	

		iv	$= 1/(\tfrac{1}{2}e^{a})$ $= (2a + 1)/2$ $[= 1/f'(a)]$ $[= 1/g'(a)]$	A1	establishing $f'(a) = 1/g'(a)$	either way round
		iv	tangents are reflections in $y = x$	B1	must mention tangents **Examiner's Comments** This proved to be more difficult, as intended for the final question in the paper. As with the integral, many candidates struggled to differentiate ½ (e^x – 1) correctly, and equally many omitted the '2' in the numerator of the derivative of ln(1 + 2x). Once these were established correctly the substitution of $x = a$ and establishing of $f'(a) = 1/g'(a)$ was generally done well, though sometimes the arguments using the result in part (i) were either inconclusive or done 'backwards'. The final mark proved to be elusive for most, as we needed the word 'tangent' used here to provide a geometric interpretation of the reciprocal gradients.	
			Total	19		
2			$\int_0^{\frac{\pi}{12}} \cos 3x \, dx = \left[\frac{\sin 3x}{3}\right]_0^{\frac{\pi}{12}}$ $= \frac{1}{3}\left(\sin\frac{\pi}{4} - 0\right)$ $= \frac{\sqrt{2}}{6}$ o.e.	B1(AO1.1) M1(AO1.1) A1(AO1.1) [3]	$\frac{\sin 3x}{3}$ Must be in exact form	
			Total	3		
3			DR	M1(AO		

			$\int_0^{\frac{1}{3}\pi}(3\sin 3x - \sin 3x)\,dx$	1.1a)	Integrating the difference of the functions
			$\left[-\frac{2}{3}\cos 3x\right]_0^{\frac{1}{3}\pi}$	B1(AO 1.1b)	$\int \sin 3x\,dx = -\frac{1}{3}\cos 3x$ oe soi
			$= -\frac{2}{3}(\cos\pi - \cos 0)$	B1(AO 3.1a)	Limits used or indicated on a diagram
			$= -\frac{2}{3}(-1-1) = \frac{4}{3}$ **AG**	A1(AO 2.1) [4]	Must be clearly shown
			Total	4	
4			$\frac{e^{3x}}{3}$	M1(AO 1.1)	soi
			$4e^{3x} + c$	A1(AO 1.1) [2]	
			Total	2	
5		a	DR At intersection $\cos x = \sin x$	M1(AO3.1a)	
			$\tan x = 1$	M1(AO1.1)	
				A1(AO1.1)	

			$x = \frac{1}{4}\pi,\ -\frac{3}{4}\pi$ Between intersections $\cos x - \sin x > 0$, so area = $= \int_{-\frac{3}{4}\pi}^{\frac{1}{4}\pi} (\cos x - \sin x)\,dx$	E1(AO2.4) [4]	Both values Explanation and completion		
		b	DR $\left[\sin x + \cos x\right]_{-\frac{3}{4}\pi}^{\frac{1}{4}\pi}$ $\sin\left(\frac{1}{4}\pi\right) - \sin\left(-\frac{3}{4}\pi\right) + \cos\left(\frac{1}{4}\pi\right) - \cos\left(-\frac{3}{4}\pi\right) = \frac{4}{\sqrt{2}}$ $2\sqrt{2}$	M1(AO1.1a) M1(AO2.1) A1(AO1.1) [3]	Limits not needed for method mark		
			Total	7			
6		a	EITHER $8\sin^2 x \cos^2 x = 2\,(1 - \cos 2x)\,(1 + \cos 2x)$ $= 2(1 - \cos^2 2x) = 2 - (1 + 2\cos 4x)$ $= 1 - \cos 4x$ OR	M1 (AO 3.1a) M1 (AO 3.1a) E1 (AO 2.1) [3]	AG Using a double angle formula Second use of a double angle formula Clearly shown		

			$8\sin^2 x\cos^2 x = 2(2\sin x\cos x)^2$	M1	Using a double angle formula	Allow any other valid sequence of identities used.
			$= 2\sin^2 2x$	M1	Another use of a double angle formula	
			$[=1 - \cos 2(2x)] = 1 - \cos 4x$	E1 [3]	Clearly shown	
			OR			
			$1 - \cos 4x = 1 - (1 - 2\sin^2 2x)$	M1	Using a double angle formula	
			$= 2\sin^2 2x$			
			$= 2(2\sin x\cos x)^2$	M1	Another use of a double angle formula	
			$= 8\sin^2 x\cos^2 x$	E1 [3]	Clearly shown	

Examiner's Comments

This question posed considerable difficulty to many candidates. Some had success by writing both $\sin^2 x$ and $\cos^2 x$ in terms of $\cos 2x$ and multiplying out. Many then did not go on to complete the proof. The first 2 marks were credited for any two applications of the double angle formulae and the final mark only when there was a convincing proof so many candidates were credited 2 out of 3 marks.

		b	$\int \sin^2 x \cos^2 x \, dx = \frac{1}{8}\int 1 - \cos 4x \, dx$ $= \frac{1}{8}x - \frac{1}{32}\sin 4x + c$	M1 (AO 1.1a) A1 (AO 1.1b) A1 (AO 1.1b) [3]	Attempt to integrate both terms $\frac{1}{4}\sin 4x$ seen or implied All correct. Must include $+c$ Examiner's Comments Most candidates realised they needed to use the identity from part (a) although many omitted the factor of $\frac{1}{8}$. Some candidates lost the final mark for omitting the $+c$.
			Total	6	
7			$\frac{dy}{dx} = 5x(-2e^{-2x}) + 5e^{-2x}$ $e^{-2x}(-10x + 5) = 0$	M1 (AO 3.1a) A1 (AO 1.1) M1 (AO 1.1a)	For use of product rule For correct (unsimplified) derivative For equating

	A1 (AO 1.1)	derivative to zero and attempting to solve for *x*	
$x = \frac{1}{2}$	M1 (AO 3.1a)	oe	
$\text{Area} = \int_0^{\frac{1}{2}} 5x\mathrm{e}^{-2x}\,\mathrm{d}x$		Limits 0 and (their) $\frac{1}{2}$ must be seen at some stage for this mark to be awarded	
$= \left[5x\left(-\frac{1}{2}\mathrm{e}^{-2x}\right)\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}}\left(-\frac{5}{2}\mathrm{e}^{-2x}\right)\mathrm{d}x$	M1 (AO 1.1a)	For integration by parts with *u* = 5*x* and *v*′ = e^{-2x}	
	A1 (AO 1.1)	First stage all correct; ignore limits for this mark	
$= -\frac{5}{4}\mathrm{e}^{-1} + \left[-\frac{5}{4}\mathrm{e}^{-2x}\right]_0^{\frac{1}{2}}$	A1 (AO 1.1)	Integration completed correctly	
$= -\frac{5}{2}\mathrm{e}^{-1} + \frac{5}{4}$	A1 (AO 1.1)	oe, but must be	Not e.g. 0.3303…
	[9]		

					exact simplified form	
			Total	9		
8		a	DR $\frac{dy}{dx} = 4x^2$ At point of contact, gradient = 1 so $4x^2 = 1$	M1 (AO 1.1a) E1(AO 2.4) [2]	AG Convincing completion needed	
			DR $x = \frac{1}{2}$ $y = \frac{1}{2}$ $\frac{1}{2} = \frac{4}{3}\left(\frac{1}{2}\right)^3 + a$ $a = \frac{1}{3}$	B1 (AO 3.2a) B1(AO 2.2a) M1(AO 3.1a) A1(AO 1.1) [4]	Taking the positive root soi	
		b	DR When curve crosses line: $\frac{4}{3}x^3 + \frac{1}{3} = x$ $4x^3 - 3x + 1 = 0$ $(2x - 1)^2(x + 1) = 0$	M1 (AO 3.1a) M1(AO 1.1) M1(AO 2.2a) A1(AO 1.1)	oe; equation with zero on one side	FT *their* (a), or working in terms of *a*, for all M marks throughout this part

			$x = -1$ $\int_{-1}^{\frac{1}{2}}\left(\frac{4}{3}x^3+\frac{1}{3}-x\right)\mathrm{d}x$ $\left[\frac{1}{3}x^4+\frac{1}{3}x-\frac{1}{2}x^2\right]_{-1}^{\frac{1}{2}}$ $\frac{9}{16}$	M1(AO 1.1) M1(AO 1.1) A1(AO 1.1) [7]	Need not include limits Need not include limits	
			Total	13		