1. The growth of a tree is modelled by the differential equation

$$10\frac{dh}{dt} = 20 - h$$

where h is its height in metres and the time t is in years. It is assumed that the tree is grown from seed, so that $h = 0$ when $t = 0$.

i. Write down the value of h for which $\frac{dh}{dt} = 0$, and interpret this in terms of the growth of the tree.

[1]

ii. Verify that $h = 20\,(1 - e^{-0.1t})$ satisfies this differential equation and its initial condition.

[5]

The alternative differential equation

$$200\frac{dh}{dt} = 400 - h^2$$

is proposed to model the growth of the tree. As before, $h = 0$ when $t = 0$.

i. Using partial fractions, show by integration that the solution to the alternative differential equation is

$$h = \frac{20\left(1 - e^{-0.2t}\right)}{1 + e^{-0.2t}}$$

[9]

ii. What does this solution indicate about the long-term height of the tree?

[1]

iii. After a year, the tree has grown to a height of 2 m. Which model fits this information better?

[3]

2. Fig. 8.1 shows an upright cylindrical barrel containing water. The water is leaking out of a hole in the side of the barrel.

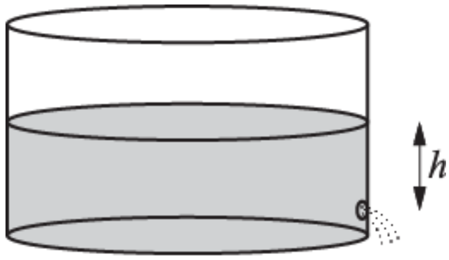


Fig. 8.1

The height of the water surface above the hole *t* seconds after opening the hole is *h* metres, where

$$\frac{dh}{dt} = -A\sqrt{h}$$

and where *A* is a positive constant. Initially the water surface is 1 metre above the hole.

i. Verify that the solution to this differential equation is

$$h = \left(1 - \tfrac{1}{2}At\right)^2$$

[3]

The water stops leaking when $h = 0$. This occurs after 20 seconds.

ii. Find the value of *A*, and the time when the height of the water surface above the hole is 0.5 m.

[4]

Fig. 8.2 shows a similar situation with a different barrel; *h* is in metres.

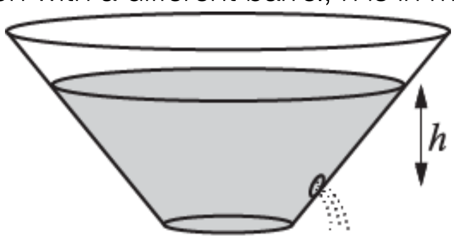


Fig. 8.2

For this barrel,

$$\frac{dh}{dt} = -B\frac{\sqrt{h}}{(1+h)^2}$$

where *B* is a positive constant. When $t = 0$, $h = 1$.

iii. Solve this differential equation, and hence show that

$$h^{\frac{1}{2}}(30+20h+6h^2) = 56-15Bt$$

[7]

iv. Given that $h = 0$ when $t = 20$, find B.

Find also the time when the height of the water surface above the hole is 0.5 m.

[4]

3. A drug is administered by an intravenous drip. The concentration, x, ofthe drug in the blood is measured as a fraction of its maximum level. The drug concentration after t hours is modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k(1+x-2x^2),$$

where $0 \le x < 1$, and k is a positive constant. Initially, $x = 0$.

i. Express $\frac{1}{(1+2x)(1-x)}$ in partial fractions.

[3]

ii. Hence solve the differential equation to show that $\frac{1+2x}{1-x} = \mathrm{e}^{3kt}$.

[7]

iii. After 1 hour the drug concentration reaches 75% of its maximum value and so $x =$ 0.75.

Find the value of k, and the time taken for the drug concentration to reach 90% of its maximum value.

[3]

iv. Rearrange the equation in part **(ii)** to show that $x = \frac{1-\mathrm{e}^{-3kt}}{1+2\mathrm{e}^{-3kt}}$.

Verify that in the long term the drug concentration approaches its maximum value.

[5]

4.

i. Show that $\frac{1}{2+x}+\frac{1}{2-x}=\frac{4}{(2+x)(2-x)}$.

[1]

In a chemical reaction, the time *t* minutes taken for a mass *x* mg of a substance to be produced is modelled by the equation

$$t=\ln\left(\frac{2+x}{2-x}\right).$$

[2]

ii. Show that when $t = 0$, $x = 0$.

[2]

iii. Show that the rate of change of *x* is proportional to the product of $(2 + x)$ and $(2 - x)$, and find the constant of proportionality.

[4]

iv. Show that $x=\frac{2\left(1-e^{-t}\right)}{1+e^{-t}}$.

Hence determine the long-term mass of the substance predicted by this model.

[4]

In another chemical reaction, the mass *x* mg at time *t* minutes is modelled by the differential equation

$$\frac{dx}{dt}=k(2+x)(2-x)e^{-t},$$

where *k* is a positive constant, and $x = 0$ when $t = 0$.

v. Show by integration that, for this reaction, $\ln\left(\frac{2+x}{2-x}\right)=4k\left(1-e^{-t}\right)$.

[5]

vi. Given that the long-term mass of this substance is 1.85 mg, find the value of *k*.

[2]

5. In a chemical reaction, the mass m grams of a chemical at time t minutes is modelled by the differential equation

$$\frac{dm}{dt} = \frac{m}{t(1+2t)}.$$

At time 1 minute, the mass of the chemical is 1 gram.

(a) Solve the differential equation to show that $m = \frac{3t}{(1+2t)}$. [8]

(b) Hence

(i) find the time when the mass is 1.25 grams, [2]

(ii) show what happens to the mass of the chemical as t becomes large. [2]

6. (a) Solve the differential equation

$$\frac{dy}{dx} = y(1+y)(1-x),$$

given that $y = 1$ when $x = 1$. Give your answer in the form $y = f(x)$, where f is a function to be determined. [9]

(b) By considering the sign of $\frac{dy}{dx}$ near (1, 1), or otherwise, show that this point is a maximum point on the curve $y = f(x)$. [3]

7. The number of bacteria in a population at time t is denoted by P. The rate of increase of P is proportional to the square root of P.

(i) Write down a differential equation relating P, the time t, and a constant of proportionality k. [1]

(ii) Verify that $P = (A + Bt)^2$, where A and B are constants, satisfies the differential equation, and find k in terms of B. [3]

8. In a chemical reaction, compound B is formed from compound A and other compounds. The mass of B at time t minutes is x kg. The total mass of A and B is always 1 kg. Sadiq formulates a simple model for the reaction in which the rate at which the mass of B increases is proportional to the product of the masses of A and B.

(a) Show that the model can be written as $\frac{\mathrm{d}x}{\mathrm{d}t} = kx(1-x)$ where k is a constant. [1]

Initially, the mass of B is 0.2 kg.

(b) Solve the differential equation, expressing x in terms of k and t. [7]

After 15 minutes, the mass of B is measured to be 0.9 kg.

(c) Find the value of k, correct to 3 significant figures. [2]

(d) Find the mass of B after 30 minutes. [1]

(e) Explain what the model predicts for the mass of A remaining for large values of t. [1]

9.

(a) Express $\frac{(x^2-8x+9)}{(x+1)(x-2)^2}$ in partial fractions. [5]

(b) Express y in terms of x given that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(x^2-8x+9)}{(x+1)(x-2)^2}$$ and $y = 16$ when $x = 3$.

[7]

10.

(a) (see insert for practice3 H640/03) The differential equation $\frac{\mathrm{d}P}{\mathrm{d}Q} = \frac{1}{k}\frac{P}{Q}$ is given on line 42. Find the general solution, giving Q as a function of P. [3]

(b) (see insert for practice3 H640/03) Hence show that, when the PED is constant, a 5% increase in price results in the demand changing by a percentage which is independent of the original price, as stated in lines 43–44. [3]

11. When a container is filled with water to a depth of y cm, the volume of water, V cm^3, in the container is modelled by the formula

$$V = 20y + 3y^2 - 0.1y^3.$$

When the container is filled to its maximum possible depth of 10 cm, the volume of water in the container
is 400 cm^3.

Water is poured into the container after t seconds at a rate which is modelled by the differential equation

$$\frac{dV}{dt} = \frac{k}{\sqrt{t+1}},$$

where k is a positive constant. Initially the container is empty and after 3 seconds it is full.

(a) Express V in terms of t. [5]

(b) Calculate the time T taken, according to the model, until the container is filled to a depth of 5 cm. [2]

(c) Calculate the rate at which the depth of water in the container is increasing at this time T. [3]

A student states that, according to the model, the water will overflow.

(d) Determine whether the student's statement is correct. [2]

12. Solve the differential equation $5x\frac{dy}{dx} = y^2 - y - 6$ given that $y = 8$ when $x = 1$. Give your answer in the form $y = f(x)$. [10]

END OF QUESTION paper

Mark scheme

Question			Answer/Indicative content	Marks	Guidance
1		i	$h = 20$, stops growing	B1	**AG** need interpretation **Examiner's Comments** Most candidates correctly wrote down the value of h but quite a number failed to give the interpretation that the tree stopped growing when its height was 20m.
		ii	$h = 20 - 20e^{-t/10}$ $dh/dt = 2e^{-t/10}$	M1A1	differentiation (for M1 need $ke^{-t/10}$, k const)
		ii	$20e^{-t/10} = 20 - 20(1 - e^{-t/10}) = 20 - h$ $= 10dh/dt$	M1	
		ii		A1	oe eg $20 - h = 20 - 20(1 - e^{-t/10}) = 20e^{-t/10}$ $= 10dh/dt$ (showing sides equivalent)
		ii	when $t = 0$, $h = 20(1 - 1) = 0$	B1	initial conditions
		ii	.... **OR** verifying by integration		....
		ii	$\int \frac{dh}{20-h} = \int \frac{dt}{10}$	M1	sep correctly and intending to integrate
		ii	$\Rightarrow -\ln(20 - h) = 0.1t + c$	A1	correct result (condone omission of c, although no further marks are possible) condone ln (h – 20) as part of the solution at this stage
		ii	$h = 0, t = 0, \Rightarrow c = -\ln 20$ $\Rightarrow \ln(20 - h) = -0.1t + \ln 20$ $\Rightarrow \ln(\frac{20-h}{20}) = -0.1t$	B1	constant found from expression of correct form (at any stage) but B0 if say $c = \ln(-20)$ (found using $\ln(h - 20)$)
		ii	$\Rightarrow 20 - h = 20e^{-0.1t}$	M1	combining logs and anti-logging (correct rules)

		ii	$\Rightarrow h = 20(1 - e^{-0.1t})$	A1	correct form (do not award if B0 above) **Examiner's Comments** Those who approached the verification by integration were quite successful. The common errors were:- • omitting the negative sign when integrating $1/(20-h)$ • omitting the constant of integration • giving $\ln(h-20)$ in their answers (without modulus signs) despite having usually given $h=20$ as a maximum value in (i) • incorrect anti-logging. Those who approached from differentiation usually obtained some marks, particularly the mark for checking the initial conditions but many gave insufficient detail when verifying the given result.
		iii	$\frac{200}{(20+h)(20-h)} = \frac{A}{20+h} + \frac{B}{20-h}$ $\Rightarrow 200 = A(20 - h) + B(20 + h)$	M1	cover up, substitution or equating coeffs
		iii	$h = 20 \Rightarrow 200 = 40B, B = 5$	A1	
		iii	$h = -20 \Rightarrow 200 = 40A, A = 5$ $200\, dh/dt = 400 - h^2$	A1	
		iii	$\Rightarrow \quad \int \frac{200}{400-h^2} dh = \int dt$	M1	separating variables and intending to integrate (condone sign error)
		iii	$\Rightarrow \quad \int (\frac{5}{20+h} + \frac{5}{20-h}) dh = \int dt$		substituting partial fractions
		iii	$\Rightarrow 5\ln(20 + h) - 5\ln(20 - h) = t + c$	A1	ft their A, B, condone absence of c, Do not allow ln (h-20) for A1.
		iii	When $t = 0$, $h = 0 \Rightarrow 0 = 0 + c \Rightarrow c = 0$	B1	cao need to **show** this. c can be found at any stage. **NB $c = \ln(-1)$** (from ln (h – 20)) or similar **scores B0.**

		iii	$\Rightarrow \quad 5\ln\frac{20+h}{20-h}=t$		
		iii	$\Rightarrow \quad \frac{20+h}{20-h}=e^{t/5}$	M1	anti-logging an equation of the correct form . Allow if $c = 0$ clearly stated (provided that $c = 0$) even if B mark is not awarded, but do not allow if c omitted. Can ft their c.
		iii	$\Rightarrow 20 + h = (20 - h)e^{t/5} = 20\,e^{t/5} - h\,e^{t/5}$ $\Rightarrow h + h\,e^{t/5} = 20\,e^{t/5} - 20$ $\Rightarrow h(e^{t/5} + 1) = 20(e^{t/5} - 1)$	DM1	making h the subject, dependent on previous mark
		iii	$\Rightarrow \quad h=\frac{20(e^{t/5}-1)}{e^{t/5}+1}$		**NB** method marks can be in either order, in which case the dependence is the other way around.(In which case, $20 + h$ is divided by $20 - h$ first to isolate h).
		iii	$\Rightarrow \quad h=\frac{20(1-e^{-t/5})}{1+e^{-t/5}}$ *	A1	**AG must have obtained B1** (for c) in order to obtain final A1. **Examiner's Comments** There were a few completely correct solutions to this part. However, many different errors were seen from the majority of candidates. There was also a lot of confused work. Those who started with the correct partial fractions, from 200/(20−h)(20+h) or 1/(20−h)(20+h), usually obtained the first three marks and then integrated having scored M1A1A1M1 thus far. Common errors then included omitting the negative sign when integrating 5/(20−h) (ie giving 5ln(20−h) and hence A0) or failing to state and then evaluate a constant. Those who had no constant were unable to score further marks. Those who did score the first 5 or 6 marks (dependent upon when the constant was evaluated) often used the laws of logarithms correctly and anti-logged although some fiddled the signs when subsequently making h the subject. Some candidates thought that 1/(400−h^2)=1/(h−20)(h+20). Marks were scored for using partial fractions on 1/(h−20)(h+20) but logarithms such as ln (h−20) for h<20 and constants such as ln(−1) could not obtain accuracy marks although the marks for anti-logging and making h the subject were still available. There were also a number who felt that 1/(400−h^2)=1/(200−h)(200+h). The use of modulus signs was rarely seen.

		iv	As $t \to \infty$, $h \to 20$. So long-term height is 20m.	B1	www **Examiner's Comments** Usually correct.
		v	1st model $h = 20(1 - e^{-0.1}) = 1.90..$	B1	Or 1st model $h = 2$ gives $t = 1.05..$
		v	2nd model $h = 20(e^{1/5} - 1)/(e^{1/5} + 1) = 1.99..$	B1	2nd model $h = 2$ gives $t = 1.003..$
		v	so 2nd model fits data better	B1 dep	dep previous B1s correct **Examiner's Comments** Most candidates scored all three marks.
			Total	19	
2		i	Either $h = (1 - ½ At)^2 \Rightarrow dh/dt = -A(1 - ½ At)$	M1	Including function of a function, need to see middle step
		i	$= -A\sqrt{h}$	A1	AG
		i	when $t = 0$, $h = (1 - 0)^2 = 1$ as required	B1	
		i	Or $\int \frac{dh}{\sqrt{h}} = \int -A\,dt$	M1	Separating variables correctly and integrating
		i	$2h^{1/2} = -At + c$	A1	Including c. [Condone change of c.]
		i	$h = \left(\frac{-At + c}{2}\right)^2$ at $t = 0$, $h = 1$, $1 = (c/2)^2 \Rightarrow c$ $= 2$, $h = (1 - At/2)^2$		
		i		B1	Using initial conditions AG

					Examiner's Comments The method of separating the variables and integrating was more popular than verification, and was more successful. Those that verified usually forgot to use the initial conditions. When integrating there was sometimes confused work when the arbitrary constant was changed but continued to be used as c.
		ii	When $t = 20$, $h = 0$	M1	Subst and solve for A
		ii	$\Rightarrow 1 - 10A = 0$, $A = 0.1$	A1	cao
		ii	When the depth is 0.5 m, $0.5 = (1 - 0.05t)^2$	M1	substitute $h = 0.5$ and their A and solve for t
		ii	$\Rightarrow 1 - 0.05t = \sqrt{0.5}$, $t = (1 - \sqrt{0.5})/0.05 = 5.86$s	A1	www cao accept 5.9 **Examiner's Comments** Good marks were scored in this part by all candidates. Some made the question more difficult when finding A by using a quadratic equation. The most common error was in using $\sqrt{0.5}$ as 0.25 when finding t.
		iii	$\frac{dh}{dt} = -B\frac{\sqrt{h}}{(1+h)^2}$ $\Rightarrow \int \frac{(1+h)^2}{\sqrt{h}}\,dh = -\int B\,dt$	M1	separating variables correctly and intend to integrate **both sides** (may appear later) [**NB reading** (1 + h)^2 **as** 1 + h^2 **eases the question. Do not mark as a MR**] In cases where $(1 + h)^2$ is MR as $1 + h^2$ or incorrectly expanded, as say $1 + h + h^2$ or $1 + h^2$, allow first M1 for correct separation and attempt to integrate and can then score a max of M1M0A0A0A1 (for $-Bt + c$) A0A0, max 2/7.
		iii	**Either**, LHS		
		iii	$\int \frac{1 + 2h + h^2}{\sqrt{h}}\,dh$	M1	expanding $(1 + h)^2$ and dividing by $\sqrt{h}$ to form a one line function of h (indep of first M1) with each term expressed as a single power of h eg must simplify say $1/\sqrt{h} + 2h/\sqrt{h} + h^2\sqrt{h}$,condone a single error for M1 (do not need to see integral signs)
		iii	$= \int (h^{-1/2} + 2h^{1/2} + h^{3/2})dh$	A1	$h^{-1/2} + 2h^{1/2} + h^{3/2}$ cao dep on second M only -do not need integral signs

		iii	Or, LHS either,	M1	
		iii	$(1 + 2h + h^2)2h^{1/2} - \int 2h^{1/2}(2 + 2h)dh$		using $\int udv + uv + \int vdu$ correct formula used correctly, indep of first M1 condone a single error for M1 if intention clear
		iii	or, $h^{1/2} + h^{3/2} + \frac{h^{5/2}}{3} + \int \frac{1}{2}h^{-3/2}(h + h^2 + \frac{h^3}{3})dh$	A1	cao oe
		iii	$2h^{1/2} + \frac{4h^{3/2}}{3} + \frac{2h^{5/2}}{5}$	A1	cao **oe, both sides dependent on first M1 mark**
		iii	$= -Bt + c$	A1	cao need $-Bt$ and c for second A1 but the constant may be on either side
		iii	$\Rightarrow 2h^{1/2} + 4h^{3/2}/3 + 2h^{5/2}/5 = -Bt + c$		
		iii	When $t = 0$, $h = 1 \Rightarrow c = 56/15$	A1	from correct work only (accept 3.73 or rounded answers here but not for final A1) or c = -56/15 if constant on opposite side.
		iii	$\Rightarrow h^{1/2}(30 + 20h + 6h^2) = 56 - 15Bt^*$	A1	NB **AG must be from all correct exact work including exact** c. **Examiner's Comments** A pleasing number of candidates scored full marks here. Most separated the variables correctly and successfully integrated the RHS, including the inclusion of $+c$. Those candidates who realised to expand the bracket and divide often were able to score all the remaining marks. A few used the approach from integration by parts but usually did not reach the end.
		iv	$h = 0$ when $t = 20$	M1	Substituting $h = 0, t = 20$
		iv	$\Rightarrow$ B = 56/300 = 0.187	A1	Accept 0.187
		iv	When $h = 0.5$ 56 − 2.8t = 29.3449…	M1	Subst their $h = 0.5$, ft their B and attempt to solve
		iv	$\Rightarrow t = 9.52$s	A1	Accept answers that round to 9.5s www. **Examiner's Comments**

					Many good scores were achieved here when substituting to find *B* and *t*. There were a lot of numerical errors from others.
			Total	18	
3		i	$\frac{1}{(1+2x)(1-x)}=\frac{A}{1+2x}+\frac{B}{1-x}\Rightarrow 1=A(1-x)+B(1+2x)$	Enter text here.	Enter text here.
		i	Enter text here.	M1	Cover up, substitution or equating coefficients
		i	$x = 1 \Rightarrow 3B = 1$, $B = 1/3$	A1	Enter text here.
		i	$x = -½ \Rightarrow 1 = 3A/2$, $A = 2/3$	A1	isw after correct A and B stated **Examiner's Comments** Was answered extremely well with nearly all candidates correctly expressing $\frac{1}{(1+2x)(1-x)}$ in partial fractions.
		ii	$1 + x - 2x^2 = (1 + 2x)(1 - x)$	B1	May be seen in separation of variables (may be implied by later working) – implied by the use of factors $(1 + 2x)$ and $(1 - x)$
		ii	$\Rightarrow \quad \frac{1}{3}\int[\frac{2}{(1+2x)}+\frac{1}{1-x}]dx=\int k dt$	M1	Separating variables and substituting partial fractions. If no subsequent work integral signs needed, but allow omission of d*x* or d*t*, but must be correctly placed if present
		ii	$\lambda \ln(1 + 2x) + \mu \ln(1 - x) = kt (+ c)$	A1	Any non-zero constant λ, μ
		ii	$\Rightarrow \ln(1 + 2x) - \ln(1 - x) = 3kt (+ c)$	A1	www oe (condone absence of *c*)
		ii	When $t = 0$, $x = 0 \Rightarrow c = 0$	B1	cao (**must** follow previous A1) need to **show** (at some stage) that $c = 0$. As a minimum $t = 0$, $x = 0$, $c = 0$. Note that $c = \ln(-1)$ (usually from incorrect integration of $(1 - x)$) or similar scores B0
		ii	$\Rightarrow \quad \ln\left(\frac{1+2x}{1-x}\right)=3kt$	M1	Combining both their log terms correctly. Follow through their c. Allow if $c = 0$ clearly stated (provided that $c = 0$) even if B mark is not awarded, but do not allow if *c* omitted

		ii	$\Rightarrow \quad \dfrac{1+2x}{1-x} = e^{3kt}$ *	A1	**AG** www must have obtained all previous marks in this part **Examiner's Comments** In part (ii) the majority of candidates were able to separate the variables and substitute their partial fractions correctly. There were, however, frequent errors in the integration usually when candidates forgot to divide by 2 when integrating $\dfrac{1}{1+2x}$ and/or when they forgot to do the same process with the −1 when integrating $\dfrac{1}{1-x}$. Many candidates did not include a constant of integration or, if included, it was either subsequently ignored or set to zero without any mathematical justification. Most candidates were able to combine theirlogarithmic terms correctly, though examiners noted the high volume of cases in which the ‘correct’ printed answer was seen following earlier incorrect working.
		iii	$(1 + 2(0.75)) / (1 - 0.75) = e^{3k}$	M1	substituting $t = 1$, $x = 0.75$ at any stage
		iii	$k = (1/3)\ln 10$ (= 0.768 (3 s.f.))	A1	3sf or better
		iii	$t = \ln(2.8/0.1)/3k = 1.45$ hours	A1	1.45 (or better) or 1 hr 27 mins **Examiner's Comments** Most candidates achieved the first two marks in part (iii) for finding the valu of k , but many made errors in handling the logarithms to find the time taken for the drug concentration to reach 90% of its maximum value. Examiners noted the large variation in the accuracy of candidates‘final answers in this part.
		iv	$1 + 2x = e^{3kt} - xe^{3kt}$	Enter text here.	Enter text here.
		iv	$\Rightarrow 2x + xe^{3kt} = e^{3kt} - 1$	M1*	Multiplying out and collecting x terms (condone one error)
		iv	$\Rightarrow x(2 + e^{3kt}) = e^{3kt} - 1$	M1dep*	Factorising their x terms correctly
		iv	$\Rightarrow x = (e^{3kt} - 1) / (2 + e^{3kt})$	A1	Enter text here.

		iv	$= (1 - e^{-3kt})/(1 + 2e^{-3kt})$*	A1	www (**AG**) – as AG must be an indication of how previous line leads to the required result (eg stating or showing multiplying by e^{-3kt})
		iv	when $t \to \infty$ $e^{-3kt} \to 0$ $x = (1 - e^{-3kt})/(1 + 2e^{-3kt}) \to 1/1 = 1$	B1	clear indication that $e^{-3kt} \to 0$ so, for example, accept as a minimum $(x \to)\dfrac{1-0}{1+0} = 1$ **or** $e^{-3kt} \to 0 \Rightarrow (x \to)\ 1$ (NB substitution of large values of t with no further explanation is B0)
		iv	OR $\dfrac{1-x}{1+2x} = e^{-3kt}$	B1	Enter text here.
		iv	$1 - x = e^{-3kt} + 2xe^{-3kt}$	M1*	Multiplying up and expanding (condone one error)
		iv	$x(1 + e^{-3kt}) = 1 - e^{-3kt}$	M1dep*	Factorising their x terms correctly
		iv	$x = (1 - e^{-3kt})/(1 + 2e^{-3kt})$*	A1	www (**AG**) – final B mark as in scheme above **Examiner's Comments** In part (iv) most candidates multiplied up by $1-x$, collected and factorised the x terms correctly. The main problem seemed to be how to get the negative exponent. These often appeared when candidates divided $e^{3kt} - 1$ by $2 + e^{3kt}$, losing the final accuracy mark in the process. It was also common for candidates to simply not show how $\dfrac{e^{3kt}-1}{2+e^{3kt}}$ was equal to the given answer. Those candidates who started by taking the reciprocal of the answer given in part(ii) part (ii) were usually far more successful in deriving the required result in this part. The majority of candidates who attempted to verify that the drug concentration approached its maximum value in the long term recognised that as $t \to \infty, e^{-3kt} \to 0$ although some candidates simply substituted a large value of t to show that x was close to 1, this approach was not sufficient to earn the final mark in this part.
			Total	18	

4		i		B1	NB AG - must be at least one intermediate step before given answer - correct application of partial fractions is fine Examiner's Comments Nearly all candidates correctly showed the required result in part (i) although a few attempted to use partial fractions with varying degrees of success. In part (ii) many candidates incorrectly verified that when $x = 0$, $t = 0$ rather than the required result of showing that when $t = 0$, $x = 0$.Those that did begin by setting $t = 0$ usually went on to score both marks in this part. Part (iii) proved to be quite discriminating with many candidates unable to show that the rate of change of x was proportional to the given product. The most common method seen was to write t as ln(2 + x) – ln(2 – x) and then to differentiate this expression with respect to x and obtain $\frac{dt}{dx} = \frac{1}{2+x} + \frac{1}{2-x}$ and then use part (i) to show that $\frac{dt}{dx} = \frac{4}{(2+x)(2-x)} \Rightarrow \frac{dx}{dt} = \frac{(2+x)(2-x)}{4}$,and hence the constant of proportionality is clearly $\frac{1}{4}$. The most common error was a failure to differentiate t correctly with many retaining the negative between the two terms. A number of candidates attempted instead to derive the given result by starting with the differential equation $\frac{dx}{dt} = k(2+x)(2-x)$ and attempting to solve this using the method of separation of variables. While a number were successful in obtaining the required constant many failed to deal with $\int \frac{dx}{(2+x)(2-x)}$ correctly or forgot to include the required constant of integration.

					Part (iv) was answered well with many correctly starting by either writing $\mathrm{e}^{t} = \frac{2+x}{2-x}$ or $\mathrm{e}^{-t} = \frac{2-x}{2+x}$, , the latter of these two usually lead to the correct given answer while the former lead to $x = \frac{2\left(\mathrm{e}^{t}-1\right)}{1+\mathrm{e}^{t}}$ with the vast majority of candidates being unable to explain clearly why this would lead to the given result. As this was a show that question there needed to be a clear indication of how this result would leads to $x = \frac{2(1-\mathrm{e}^{-t})}{1+\mathrm{e}^{-t}}$. Finally in this part many candidates correctly stated that the long-term mass of the substance was 2 mg. Part (v) was answered with varying degrees of success with the vast majority correctly separating the variables to obtain $\int \frac{\mathrm{d}x}{(2-x)(2+x)} = k\int \mathrm{e}^{-t}\mathrm{d}t$ - however, from this point it was all too clear that a number of candidates did not, as requested, show by integration the given result, but simply wrote down the given answer (or an answer only a single step away from the given answer) without clearly showing how either side of the given equation was obtained. In many cases candidates failed to include a constant of integration that needed to be found using the given initial conditions. Part (vi) was answered extremely well with many candidates obtaining the correct answer of 0.811 which was achieved by setting e^{-t} equal to zero and substituting 1.85 for *x*. The most common error seen by examiners was to set $\ln\left(\frac{2+x}{2-x}\right)$ equal to 1.85 and solve for *k* with e^{-t} equal to zero.

		ii		B1	or = e^0 or $\ln(2 + x) = \ln(2 - x)$
		ii	$2 + x = 2 - x \Rightarrow x = 0$	B1	If only this line seen then award B0B1 **SC: Allow B1 only for verifying that when $x = 0$, $t = 0$**
		iii		B1	
		iii		M1	**Correct** differentiation of their t **OR for first two marks** - If no subtraction law of logs seen e.g. $\frac{dt}{dx} = \left(\frac{1}{\left(\frac{2+x}{2-x}\right)}\right)\left(\frac{(2-x)(1)-(2+x)(-1)}{(2-x)^2}\right)$ award B1 for correct first bracket (reciprocal expression) and B1 for second correct bracket (quotient/chain rule)(oe) – if additional constant(s) added (e.g. $t = k\ln(\ldots)$) then award B1 only for a constant times a fully correct derivative
		iii		A1	$\frac{dt}{dx}$ and $\frac{dx}{dt}$ must be correctly attributed to the correct expression for this mark
		iii	$k = \frac{1}{4}$ See next page for an alternative solution	A1	Explicitly stating (that the constant of proportionality is) $\frac{1}{4}$ - therefore it is possible to score A0A1in this part
		iii	OR $\frac{dx}{dt} = k(2+x)(2-x) \Rightarrow \int \frac{dx}{(2+x)(2-x)} = k\int dt$	B1	Allow omission of dx and/or dt
		iii	$\lambda \ln(2 + x) + \mu \ln(2 - x) = kt(+c)$	M1	Any non-zero constant λ, μ - further guidance in (v) for this and the next mark

		iii	$\frac{1}{4}\left[\ln(2+x)-\ln(2-x)\right]=kt(+c)$ $x=0,\ t=0 \Rightarrow c=0$	A1	www oe (condone absence of c)
		iii		A1	www - must include c and show that $c = 0$. Must explicitly state that $k=\frac{1}{4}$
		iv	$e^t=\frac{2+x}{2-x}$ $(2-x)e^t = 2 + x$	B1	
		iv	$\Rightarrow 2e^t - xe^t = 2 + x$ $\Rightarrow x(1 + e^t) = 2e^t - 2$	M1	Multiplying out, collecting x terms (condone sign slips and numerical errors (eg loss of a 2) only but M0 if e^t incorrectly replaced with e^{-t}) **and** factorising their x terms correctly
		iv		A1	www **NB AG** – as AG must be an indication of how previous line leads to the required result (eg stating or showing multiplying by e^{-t})
		iv		B1	
		iv	**OR (for first three marks)** $e^{-t}=\frac{2-x}{2+x}$ $(2+x)e^{-t} = 2 - x$	B1	
		iv	$\Rightarrow 2e^{-t} + xe^{-t} = 2 - x$ $\Rightarrow x(1 + e^{-t}) = 2 - 2e^{-t}$	M1	Multiplying out, collecting x terms (condone sign slips as above) and factorising their x terms correctly
		iv		A1	www **NB AG**
		v		M1*	Separating variables - condone sign slips and issues with placement of k but M0 for $\int(2-x)(2+x)\mathrm{d}x =$… or equivalent algebraic error in separating variables unless recovered. If no subsequent work integral signs needed, but allow omission of dx and/or dt but must be correctly placed if present
		v	$\alpha \ln(2 + x) + \beta \ln(2 - x) = \gamma e^{-t}(+c)$	A1	Any non-zero constants α, β, γ - **this line must be seen and cannot be implied by later working (as this is an AG)** – condone absence of c or if a constant present condone the use of k for their

					constant. Do not condone invisible brackets e.g. ln 2 + x unless recovered before subtraction law of logs applied – all of these points apply to the next A mark too
		v	$\ln(2+x) - \ln(2-x) = -4ke^{-t}(+c)$	A1	www oe
		v	When $t = 0$, $x = 0 \Rightarrow c = 4k$	M1dep*	Substituting $x = 0$, $t = 0$ into each term in an attempt to find their c (must get $c = \ldots$) - if they integrate and use k as their constant they must use $x = 0$, $t = 0$ to find this single k term only
		v		A1	www **NB AG** must have obtained all previous marks in this part
		v	OR (for first 3 marks) – final M1A1 as above $\int \frac{1}{(2-x)(2+x)}\mathrm{d}x = k\int e^{-t}\mathrm{d}t$	M1*	Separating variables. If no subsequent work integral signs needed, but allow omission of dx or dt, but must be correctly placed if present
		v		A2	**Must see** $1/(4 - x^2)$ on lhs – please note that one A mark cannot be awarded
		vi	as $t \to \infty$, $x \to 1.85 \Rightarrow \ln 3.85/0.15 = 4k$	M1	Sets e^{-t} to 0 and substitutes $x = 1.85$ – condone substitution of a 'large' value of t only if it leads to the correct value of k
		vi	$\Rightarrow k = 0.811$	A1	$k = 0.25 \ln(77/3)$ or 0.81 or better
			Total	18	
5		a	$\int \frac{\mathrm{d}m}{m} = \int \frac{\mathrm{d}t}{t(1+2t)}$ $\frac{1}{t(1+2t)} \equiv \frac{A}{t} + \frac{B}{1+2t}$ $\Rightarrow 1 \equiv A(1+2t) + Bt$ $t = 0 \Rightarrow A = 1$ $t = -½ \Rightarrow 1 = -½B \Rightarrow B = -2$ $\Rightarrow \int \frac{\mathrm{d}m}{m} = \int \left(\frac{1}{t} - \frac{2}{1+2t}\right)\mathrm{d}t$ $\Rightarrow \ln m = \ln t - \ln(1+2t) + c$ $t = 1$, $m = 1 \Rightarrow c = \ln 3$	M1(AO1.1a) M1(AO3.1b) M1(AO1.1) A1A1(AO1.1 1.1) B1FT(AO2.1) M1(AO1.1) E1(AO2.1) [8]	separating variables using partial fractions substituting values, equating coeffs or cover up $A = 1$, $B = -2$ FT their *i*, *ii*, condone no c evaluating constant of integration

			$\Rightarrow \ln m = \ln\left(\frac{3t}{1+2t}\right)$ $\Rightarrow m = \frac{3t}{(1 + 2t)}$		AG
	b	i	$1.25 = \frac{3t}{(1+2t)}$ $\Rightarrow 1.25 + 2.5t = 3t$ $\Rightarrow t = 1.25 \div 0.5 = 2.5$ minutes	M1(AO1.1a) A1(AO1.1) [2]	Enter text here.
	b	ii	$m = \frac{3}{\left(\frac{1}{t} + 2\right)}$ $\rightarrow 1.5$ [grams]	M1(AO3.1b) A1(AO2.2a) [2]	Enter text here.
			Total	12	
6	a		$\int \frac{1}{y(1+y)} \, dy = \int (1-x) \, dx$ $\frac{1}{y(1+y)} = \frac{1}{y} - \frac{1}{1+y}$	M1(AO 1.1a) M1(AO 3.1a) A1(AO 1.1b) A1(AO 1.1b) M1(AO 1.1a)	Separating the variables Correct form $\frac{A}{y} + \frac{B}{1+y}$ used Correct partial fractions

			$\ln y - \ln(1+y) = x - \frac{1}{2}x^2 + c$ $\ln 1 - \ln(1+1) = 1 - \frac{1}{2} + c$ $c = -\frac{1}{2} - \ln 2$ $\ln y - \ln(1+y) = x - \frac{1}{2}x^2 - \frac{1}{2} - \ln 2$ $\ln\left(\frac{2y}{1+y}\right) = x - \frac{1}{2}x^2 - \frac{1}{2}$ $\frac{2y}{1+y} = e^{x - \frac{1}{2}x^2 - \frac{1}{2}}$ $2y = (1+y)e^{x - \frac{1}{2}x - \frac{1}{2}x^2}$ $y = \frac{e^{x - \frac{1}{2}x^2 - \frac{1}{2}}}{2 - e^{x - \frac{1}{2}x^2 - \frac{1}{2}}}$	A1(AO 1.1b) M1(AO 1.1a) M1(AO 1.1a) A1(AO 1.1b) [9]	oe, e.g. RHS $-\frac{1}{2}(1-x)^2 + c$ Use of (1, 1) to find c Writing in non-logarithmic form Making y the subject Correct $y = f(x)$	
		b	At, (1, 1), $\frac{dy}{dx} = 1 \times 2(1-1) = 0$ Near (1, 1), $y(1+y)$ is positive and $(1-x)$ is positive for $x < 1$ but negative for $x > 1$ $\frac{dy}{dx} > 0$ for $x < 1$ and $\frac{dy}{dx} < 0$ for $x > 1$ so	B1(AO 3.1a) M1(AO 1.1b) E1(AO 2.2a)	Determining sign of gradient on each side	

			maximum point Alternative method At, (1, 1), $\frac{dy}{dx} = 1\times 2(1-1) = 0$ $\frac{d^2y}{dx^2} = (1+2y)\frac{dy}{dx}(1-x) + (y+y^2)(-1)$ $\frac{d^2y}{dx^2} = 3\times 0 + 2(-1) = -2 < 0$ (1, 1) is a maximum point	B1 M1 E1 [3]	Clear deduction seen Differentiation using product rule and evaluation at (1, 1) Condone –2 not seen if clearly negative Must follow both 1st and 2nd derivatives
			Total	12	
7		i	$\frac{dP}{dt} = k\sqrt{P}$	B1 [1]	oe – condone $dP = k\sqrt{P}dt$ **Examiner's Comments** In part(i) most candidates correctly wrote down the differential equation relating *P*, the time *t*, and the constant *k*. The most common errors in this part were those candidates who wrote $\frac{dt}{dP} = k\sqrt{P}$ or $\frac{dP}{dt} = kP^2$.

		ii	$P=(A+Bt)^2 \Rightarrow \dfrac{dP}{dt}=2(A+Bt)B$ $\dfrac{dP}{dt}=2B\sqrt{P}$ $k=2B$ OR $\int \dfrac{dP}{\sqrt{p}}=k\int dt \Rightarrow 2\sqrt{P}=kt(+c)$ $P=\left(\dfrac{1}{2}kt+A\right)^2$	M1 A1 A1 [3] M1 A1 A1	Attempt at chain rule – allow this mark if B absent or incorrect but must include $2(A+Bt)$ – if multiplied out condone one error in differentiation oe – correctly showing that the rate of increase of P is proportional to the square root of P Not for $B=\dfrac{k}{2}$ Separates their variables correctly and attempt to integrate for theirdifferential equation given in (i) – for an attempt powers must increase by 1 (oe) but not for k – condone lack of $+ c$ Correct integration – accept any (correct) constant for A – coefficient of t maybe implicit stated e.g. $P=(Ct+A)^2$ with $k=2C$ seen Not for $B=\ldots$ Examiner's Comments Even though in part (ii) the question asked for candidates to verify that $P=(A+Bt)^2$ was a solution to the differential equation many decided instead to solve the differential equation by separating the variables and integrating. It was disappointing how many candidates gave the final answer in

			$k = 2B$		this part as $B = \frac{k}{2}$ even though the question specifically asked for k in terms of B.
			Total	4	
8		a	$\frac{dx}{dt}$ is the rate at which x is increasing Mass of B is x, so mass of A is $(1 - x)$ $\frac{dx}{dt} \propto x(1-x)$, so $\frac{dx}{dt} = kx(1-x)$	B1(AO2.1) [1]	Must indicate where terms come from AG
		b	$\int \frac{1}{x(1-x)} dx = \int k dt$ $\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$ $1 \equiv A(1-x) + Bx \Rightarrow A = 1, B = 1$ $\int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = \int k\, dt \Rightarrow \ln x - \ln(1-x) = kt + c$ $t = 0,\ x = 0.2 \Rightarrow c = \ln\frac{1}{4}$ (oe)	M1(AO3.1a) M1(AO3.1a) A1(AO2.2a) M1*(AO1.1a) M1dep*(AOs3.1a) M1dep*(AO1.1b) A1(AOs2.5)	Separation of variables Find partial fractions (may be implied) Condone sign error, but must have two ln terms and +c Use of initial conditions; may be done after equation is rearranged Rearrange equation to remove logs; may be done before finding c

			$\frac{4x}{1-x} = e^{kt}$ (oe) $x = \frac{e^{kt}}{4+e^{kt}}$	[7]	oe, but must be of the form $x = f(t)$
		c	$t = 15$, $x = 0.9$ &⇒ $36 = e^{15k}$ (oe) $k = 0.239$ to 3sf	M1(AO3.1) A1(AO1.1b) [2]	Substitute values in their solution
		d	$t = 30 \Rightarrow$ mass of B is $\frac{e^{0.239\times30}}{4+e^{0.239\times30}} = 0.997$ kg	B1(AO3.4) [1]	
		e	As $t \to$ ro, $x \to 1$ and so $1 - x \to 0$, so the model predicts there is a very small amount of A remaining when t is large	B1(AO3.5a) [1]	May evaluate x for large t (eg $t = 100$)
			Total	12	
9		a	$\frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$ $x^2 - 8x + 9 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$ $A = 2$	B1 (AO 3.1a) M1 (AO 2.1)	may be seen later

			Answer	Marks	Guidance
			$B = -1$ $C = -1$	A1 (AO 1.1) A1 (AO 1.1) A1 (AO 1.1) [5]	$\frac{2}{(x+1)} - \frac{1}{(x-2)} - \frac{1}{(x-2)^2}$ **Examiner's Comments** Candidates who did well recognised the correct form of partial fractions and were able to work successfully to find the coefficients. Candidates who did less well made algebraic slips in clearing the fractions or made slips in arithmetic when finding the coefficients.
	b		$\int \frac{dy}{y} = \int \frac{x^2 - 8x + 9}{(x+1)(x-2)^2} dx$ soi use of their partial fractions in integration $\ln\|y\| = 2\ln\|x+1\| - \ln\|x-2\| + \frac{1}{x-2} + c$	M1* (AO 3.1a) M1* (AO 2.1) A1 (AO 1.1) A1 (AO 1.1)	allow omission of integral signs and/or omission of d*y* and/or d*x* allow one sign error and/or one coefficient error A1 for any correct natural log integral on RHS FT *their* $\frac{2}{x+1}$ or *their* $\frac{-1}{x-2}$ condone use of brackets instead of modulus signs; these two **A** marks are only available following the award of **both M** marks

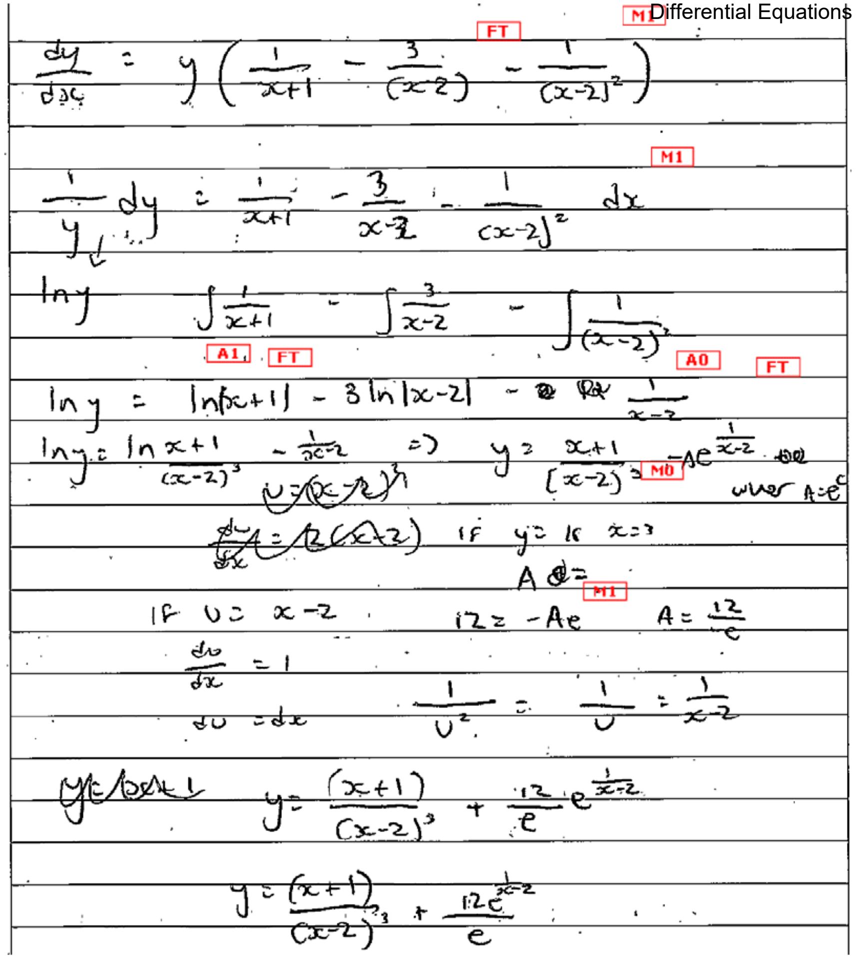
substitution of $y = 16$ and $x = 3$	M1dep* (AO 1.1)	A1 for $\frac{1}{x-2}$ FT *their* $\frac{k}{(x-2)^2}$	may be awarded following collection of like terms, which may contain errors NB $c = -1$
correctly exponentiate both sides of their equation	M1 (AO 1.1)	expression must include + c and must include at least one natural log term; may be awarded after exponentiating	
$y = \frac{(x+1)^2}{x-2}e^{\frac{3-x}{x-2}}$ oe	A1 (AO 2.1)		
	[7]	eg $\frac{(x+1)^2}{x-2}e^{\frac{1}{x-2}}\ e^{-1}$	

Examiner's Comments

Candidates who did well recognised the need to use their result from part (a). They separated the variables successfully and were then able to integrate and substitute the values of x and y to find the constant of integration. Candidates who did very well were able to go on and find a correct expression for y.

Candidates who did less well rearranged incorrectly when they attempted to separate the variables, or were unable to integrate the quadratic term correctly. They made slips in

					exponentiating both sides of their equation, usually assuming that the operation is distributive. Exemplar 4

					 In this response FT marks have been credited for the use of their partial fractions and separation of variables. One A mark has been credited FT, but the integration of the quadratic term went astray. The exponentiation of both sides was incorrect, but in spite of this, the method mark for substitution was subsequently earned.
			Total	12	

10		a	$\int \frac{1}{P} \mathrm{d}P = \frac{1}{k} \int \frac{1}{Q} \mathrm{d}Q$ $\ln P = \frac{1}{k} \ln Q \ [+c]$ $Q = AP^k$ [where A is a constant]	M1 (AO 1.1a) M1 (AO 1.1) A1 (AO 2.5) [3]	Separation of variables Integration oe; must include constant here	
		b	$P_1 = 1.05P_0$ $Q_0 = AP_0^k$ and $Q_1 = A \times 1.05^k P_0^k$ Percentage change in Q is $100\left(\frac{A \times 1.05^k P_0^k - AP_0^k}{AP_0^k}\right)$ $= 100(1.05^k - 1)$ and so is independent of original price	M1 (AO 3.1b) M1 (AO 3.4) E1 (AO 2.1) [3]	notation may vary, e.g. just $1.05P$ oe Convincing completion	
			Total	6		
11		a	$\int \mathrm{d}V = \int \left(\frac{k}{\sqrt{t+1}}\right) \mathrm{d}t$	M1 (AO3.1a) M1 (AO1.1)	Separation of variables. Allow omission of integral signs and or dt or dV Allow one error eg omission of + c	

			$V = 2k(t+1)^{\frac{1}{2}} + c$ Substitution of both conditions k = 200 or c = −400 $V = 400\sqrt{t+1} - 400$ oe	M1 (AO2.1) M1 (AO3.3) M1 (AO1.1) [5]	$0 = 2k + c$ $400 = 4k + c$ For either of these All correct
		b	y = 5, V = 162.5 T = 0.9775(390625 …) BC	B1 (AO3.4) B1 (AO1.1) [2]	Rounded to 2, 3 or 4 sf, awrt 0.9775
		c	$\frac{dV}{dt} = \frac{dV}{dy} \times \frac{dy}{dt}$ used $\frac{200}{\sqrt{1+0.9775...}} = (20 + 6\times 5 - 0.3\times 5^2)\frac{dy}{dt}$ 3.346(40522876 …)	M1 (AO3.1a) M1 (AO2.1) A1 (AO1.1) [3]	Allow one slip in derivative Rounded to 2, 3 or 4 sf, awrt 3.35
		d	*Either* at $t = 3$, $\frac{dV}{dt} = 100\text{cm}^3\text{s}^{-1}$ or	B1 (AO3.4)	

			$\frac{dy}{dt} = 2\text{cms}^{-1}$ Therefore volume/height still increasing so the student is correct	E1 (AO3.5a) [2]	
			Total	12	
12			$\int \frac{5}{y^2 - y - 6} dy = \int \frac{1}{x} dx$ $\frac{5}{y^2 - y - 6} = \frac{A}{y-3} + \frac{B}{y+2}$ $A = 1$, $B = -1$ $\int \left(\frac{1}{y-3} - \frac{1}{y+2}\right) dy = \int \frac{1}{x} dx$ $\Rightarrow \ln(y-3) - \ln(y+2) = \ln x + c$ When $x = 1$, $y = 8$ so $\ln 5 - \ln 10 = \ln 1 + c$ $c = \ln \frac{1}{2}$ $\ln(y-3) - \ln(y+2) = \ln x + \ln \frac{1}{2}$ So $\ln\left(\frac{y-3}{y+2}\right) = \ln\left(\frac{x}{2}\right) \Rightarrow \frac{y-3}{y+2} = \frac{x}{2}$	M1 (AO 3.1a) M1 (AO 3.1a) A1 (AO 1.1b) M1 (AO 1.1a) A1 (AO 1.1b) M1 (AO 1.1b) A1 (AO 1.1b) M1 (AO 3.1a) M1 (AO 1.1a) A1 (AO 1.1b) [10]	Attempt to separate variables Attempt to use partial fractions — With their factors All correct Integrating, obtaining natural logs FT their partial fractions Use of initial conditions to find c oe Attempt to remove logs from their eqn

			$2y - = xy + 2x\ y(2 - x) = 6 + 2x$ $\Rightarrow y = \dfrac{6+2x}{2-x}$		Attempt to make y the subject oe, but must be $y = f(x)$ form	Of their equation
			Total	10		