1.

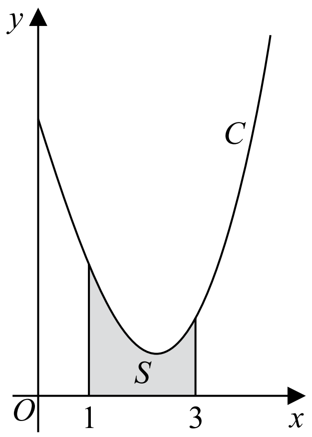


Figure 4

Figure 4 shows a sketch of part of the curve C with equation

$$y = \frac{x^2 \ln x}{3} - 2x + 5, \quad x > 0$$

The finite region S, shown shaded in Figure 4, is bounded by the curve C, the line with equation $x = 1$, the x-axis and the line with equation $x = 3$

The table below shows corresponding values of x and y with the values of y given to 4 decimal places as appropriate.

1/2 1/2 1/2 1/2

x	1	1.5	2	2.5	3
y	3	2.3041	1.9242	1.9089	2.2958
	y_0	y_1	y_2	y_3	y_4

(a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of S, giving your answer to 3 decimal places.

(3)

(b) Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of S.

(1)

(c) Show that the exact area of S can be written in the form $\frac{a}{b} + \ln c$, where a, b and c are integers to be found.

(6)

(In part c, solutions based entirely on graphical or numerical methods are not acceptable.)

a) $A = \frac{1}{2} \times h[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$ Trapezium Rule

$h = 1.5 - 1 = 2 - 1.5 = 0.5 \Rightarrow h = 0.5$ ①

$A = \frac{1}{2} \times \frac{1}{2}[(3 + 2.2958) + 2(2.3041 + 1.9242 + 1.9089)] = 4.39255$

$\Rightarrow$ Area of S is $\underline{4.393}$ (3 d.p) ①

Question continued

b) • h is the width of intervals

$\Rightarrow$ Option 1: decrease h (width of the strips) 1

Option 2: increase the number of strips

c)

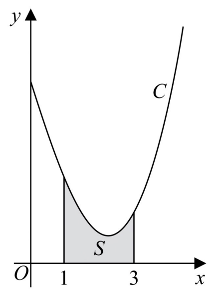


Figure 4

$$y = \frac{x^2 \ln x}{3} - 2x + 5$$

$$A = \int_1^3 \frac{x^2 \ln x}{3} - 2x + 5 \, dx$$

Integration by parts: 1

$$\int \frac{x^2 \ln x}{3} \, dx = f(x) \cdot g(x) - \int f(x) \cdot g'(x) \, dx$$

let $f'(x) = \frac{x^2}{3}$ $\quad f(x) = \frac{x^3}{9}$ $\quad \Rightarrow \int \frac{x^2 \ln x}{3} \, dx = \frac{x^3}{9} \ln(x) - \int \frac{x^3}{9} \cdot \frac{1}{x} \, dx$ 1

$g(x) = \ln(x)$ $\quad g'(x) = \frac{1}{x}$

$$= \frac{x^3}{9} \ln(x) - \frac{1}{9} \int x^2 \, dx$$

$$= \frac{x^3}{9} \ln(x) - \frac{x^3}{27} + C$$ 1

$$\Rightarrow A = \int_1^3 \frac{x^2 \ln x}{3} - 2x + 5 \, dx$$

$$\Rightarrow A = \left[\frac{x^3}{9} \ln x - \frac{x^3}{27} - x^2 + 5x \right]_1^3 = \left[\frac{3^3}{9} \ln(3) - \underbrace{\frac{3^3}{27} - 9 + 15}_{-1-9+15} \right] - \left[\underbrace{\frac{1}{9} \ln(1)}_{=0} - \underbrace{\frac{1}{27} - 1 + 5}_{\frac{107}{27}} \right]$$

1

$\ln(1) = 0$

$$\Rightarrow A = (3\ln(3) + 5) - (107/27)$$

$*$ $a \ln(b) = \ln(b^a)$

$$\Rightarrow A = 3\ln(3) + \frac{28}{27}$$

$$\Rightarrow A = \ln(27) + \frac{28}{27}$$

$a = 28$, $b = 27$ and $c = 27$. 1

2. The speed of a small jet aircraft was measured every 5 seconds, starting from the time it turned onto a runway, until the time when it left the ground.

The results are given in the table below with the time in seconds and the speed in $m\,s^{-1}$.

Time (s)	0	5	10	15	20	25
Speed ($m\,s^{-1}$)	2	5	10	18	28	42

Using all of this information,

(a) estimate the length of runway used by the jet to take off.

(3)

Given that the jet accelerated smoothly in these 25 seconds,

(b) explain whether your answer to part (a) is an underestimate or an overestimate of the length of runway used by the jet to take off.

(1)

a) length of Runway ?

Time (s)	0	5	10	15	20	25
Speed ($m\,s^{-1}$)	2 y_0	5 y_1	10 y_2	18 y_3	28 y_4	42 y_5

$$\boxed{A = \frac{1}{2} \times h\left[y_0 + y_n + 2(y_1 + \dots + y_{n-1})\right]}$$ 1

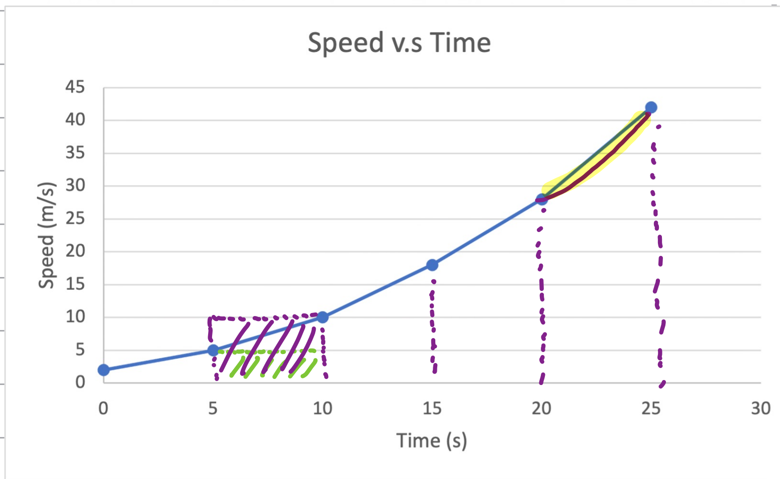
$h = 5$

1

$$A = \frac{1}{2} \times 5\left[2 + 42 + 2(5 + 10 + 18 + 28)\right]$$

$\Rightarrow A = 415m$

$\Rightarrow$ length of runway is 415m. 1



b) We used the trapezium rule, and we have a concave upwards graph, So we therefore have an overestimate, Since the top of the trapezium is over the curve. 1

3. The table below shows corresponding values of x and y for $y = \sqrt{\frac{x}{1+x}}$

The values of y are given to 4 significant figures.

x	0.5	1	1.5	2	2.5
y	0.5774	0.7071	0.7746	0.8165	0.8452

(a) Use the trapezium rule, with all the values of y in the table, to find an estimate for

$$\int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}}\,dx$$

giving your answer to 3 significant figures.

(3)

a) Trapezium Rule : $\int_{x_0}^{x_n} f(x)\,dx = \frac{1}{2}\cdot h\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\right]$

	x_0	x_1	x_2	x_3	x_4
x	0.5	1	1.5	2	2.5
y	0.5774	0.7071	0.7746	0.8165	0.8452
	y_0	y_1	y_2	y_3	y_4

0.5 0.5 0.5 0.5

What is h? h is the difference between each value of x.

$\Rightarrow h = \underline{0.5}$ (1)

$$\Rightarrow \int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}}\,dx \approx \frac{1}{2}\times 0.5\left[(0.5774 + 0.8452) + 2(0.7071 + 0.7746 + 0.8165)\right] \quad (1)$$

$$\approx 1.50475\ldots$$

$$\Rightarrow \int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}}\,dx \approx \underline{1.50} \quad (1)$$

(b) Using your answer to part (a), deduce an estimate for $\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\, dx$

(1)

b) From part a : $\int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}}\, dx \approx 1.50$

this is a constant, so we can take it out the integral.

$$\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\, dx = \int_{0.5}^{2.5} \sqrt{9}\sqrt{\frac{x}{1+x}}\, dx = 3\underbrace{\int_{0.5}^{2.5} \sqrt{\frac{x}{1+x}}\, dx}_{\text{this is the same as we had in part a}}$$

$$\Rightarrow \int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\, dx \approx 3 \times 1.50 = \underline{\underline{4.50}} \quad 1$$

Given that

$$\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\, dx = 4.535 \text{ to 4 significant figures}$$

(c) comment on the accuracy of your answer to part (b).

(1)

c) Estimate from part b : $\int_{0.5}^{2.5} \sqrt{\frac{9x}{1+x}}\, dx \approx 4.50$

The accuracy of the answer in part b is high, since $\underset{\text{estimate}}{4.50} \approx \underset{\text{real value}}{4.535}$ 1

4.

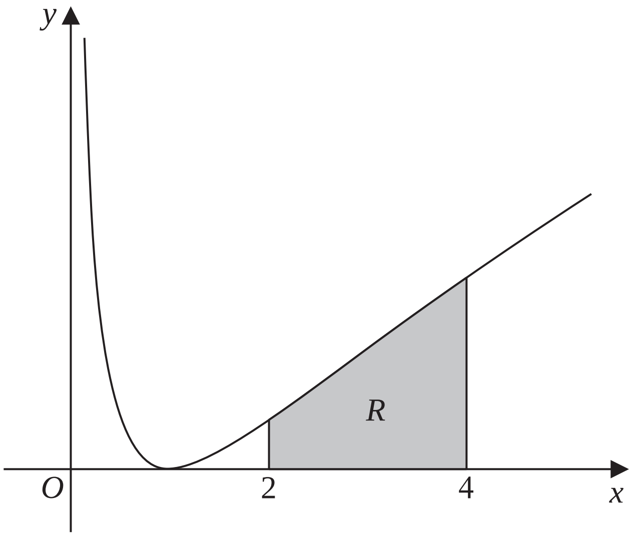


Figure 2

Figure 2 shows a sketch of part of the curve with equation

$$y = (\ln x)^2 \qquad x > 0$$

The finite region R, shown shaded in Figure 2, is bounded by the curve, the line with equation $x = 2$, the x-axis and the line with equation $x = 4$

The table below shows corresponding values of x and y, with the values of y given to 4 decimal places.

+0.5 +0.5 +0.5 +0.5 $\therefore h = 0.5$

x	2	2.5	3	3.5	4
y	0.4805	0.8396	1.2069	1.5694	1.9218

y_0 y_1 y_2 y_3 y_4

(a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of R, giving your answer to 3 significant figures.

(3)

(b) Use algebraic integration to find the exact area of R, giving your answer in the form

$$y = a(\ln 2)^2 + b\ln 2 + c$$

where a, b and c are integers to be found.

(5)

(a) $h = 0.5$ 1

$\int_2^4 (\ln x)^2\,dx \approx \frac{1}{2} \times 0.5 \times \{0.4805 + 1.9218 + 2(0.8396 + 1.2069 + 1.5694)\}$ 1

$= 2.408525$

Area of R $= 2.41$ units^2 (3sf) 1

Question continued

(b) $\int_2^4 (\ln x)^2\,dx \equiv \int_2^4 \ln x \times \ln x \; dx$

u $\frac{dv}{dx}$

Formula :

$u = \ln x \qquad v = \;?$

$\int u \frac{dv}{dx}\,dx = uv - \int v \frac{du}{dx}\,dx$

$u' = \frac{1}{x} \qquad v' = \ln x$

To find v, we integrate v' :

$\int \ln x \, dx \equiv \int \ln x \times 1 \, dx$

u v'

$u = \ln x \qquad v = x$

$u' = \frac{1}{x} \qquad v' = 1$

$\therefore \int \ln x \, dx = x \ln x - \int x \left({}^1/_x \right) dx$

$= x\ln x - x$

$\int_2^4 (\ln x)^2\,dx \equiv \int_2^4 \ln x \times \ln x \, dx$

$u = \ln x \qquad v = x\ln x - x$

$u' = \frac{1}{x} \qquad v' = \ln x$

$\therefore \int_2^4 (\ln x)^2\,dx = \ln x\,(x\ln x - x) - \int (x\ln x - x)\left(\frac{1}{x}\right) dx$ (1)

$= x\ln x\,(\ln x - 1) - \int (\ln x - 1)\,dx$

$= x\ln x\,(\ln x - 1) - \{ x\ln x - x - x \} + c$

$= x(\ln x)^2 - x\ln x - x\ln x + 2x + c$ (1)

$= x(\ln x)^2 - 2x\ln x + 2x + c$ (1)

Question continued

$$\therefore \int_2^4 (\ln x)^2\,dx = \left[x(\ln x)^2 - 2x\ln x + 2x\right]_2^4$$

$$= \{4(\ln 4)^2 - 2(4)\ln 4 + 2(4)\} - \{2(\ln 2)^2 - 2(2)\ln 2 + 2(2)\}$$

$$= \{4(\ln 2^2)^2 - 8\ln 2^2 + 8\} - \{2(\ln 2)^2 - 4\ln 2 + 4\}$$

$$= \{4(2\ln 2)^2 - 16\ln 2 + 8\} - \{2(\ln 2)^2 - 4\ln 2 + 4\}$$

$$= 16(\ln 2)^2 - 16\ln 2 + 8 - 2(\ln 2)^2 + 4\ln 2 - 4 \quad (1)$$

$$= 14(\ln 2)^2 - 12\ln 2 + 4 \quad (1)$$

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