1. Show that

$$\int_0^2 2x\sqrt{x+2}\,\mathrm{d}x = \frac{32}{15}\left(2+\sqrt{2}\right)$$

(7)

$\int_0^2 2x(x+2)^{1/2}\,dx$

let $u = x+2$

$\frac{du}{dx} = 1$

$du = dx$

$x+2 = u$

$\therefore x = u-2$

@ $x = 2,\ u = 4$

@ $x = 0,\ u = 2$

$= \int_2^4 (2u-4)(u^{1/2})\,du$

$= \int_2^4 2u^{3/2} - 4u^{1/2}\,du$

$= \left[\frac{2u^{5/2}}{5/2} - \frac{4u^{3/2}}{3/2}\right]_2^4$

$= \left[\frac{4u^{5/2}}{5} - \frac{8u^{3/2}}{3}\right]_2^4$

$= \left(\frac{4\times 4^{5/2}}{5} - \frac{8\times 4^{3/2}}{3}\right) - \left(\frac{4\times 2^{5/2}}{5} - \frac{8\times 2^{3/2}}{3}\right)$

$= \frac{128}{5} - \frac{64}{3} - \frac{16\sqrt{2}}{5} + \frac{16\sqrt{2}}{3}$

$= \frac{64+32\sqrt{2}}{15} = \frac{32(2+\sqrt{2})}{15}$

$\frac{ab}{c} = \frac{a}{c}\times b$

$= \frac{32}{15}(2+\sqrt{2})$ as required.

2. (a) Use the substitution $x = u^2 + 1$ to show that

$$\int_5^{10} \frac{3\,dx}{(x-1)\left(3+2\sqrt{x-1}\right)} = \int_p^q \frac{6\,du}{u(3+2u)}$$

where p and q are positive constants to be found.

(4)

a) $\displaystyle\int_5^{10} \frac{3}{\underbrace{(x-1)}_{u^2+1}\,(3+2\underbrace{\sqrt{x-1}}_{u})}\,\underbrace{dx}_{2\sqrt{x-1}\,du}$

Integration by Substitution :

$x = u^2 + 1$

$u^2 = x - 1 \Rightarrow u = \sqrt{x-1}$

$du = \dfrac{1}{2\sqrt{x-1}}\,dx$

$\Rightarrow dx = 2\sqrt{x-1}\,du$ ①

$= \displaystyle\int_2^3 \frac{3}{(u^2+1-1)(3+2u)} \cdot 2u\,du$ ①

$= \displaystyle\int_2^3 \frac{6u}{u^2 \cdot (3+2u)}\,du$

New Limits: For $x = 5$, $u = \sqrt{5-1} = \sqrt{4} = 2$

$x = 10$, $u = \sqrt{10-1} = \sqrt{9} = 3$ new limits ①

$= \displaystyle\int_2^3 \frac{6}{\underline{u(3+2u)}}\,du$ as required, with $\underline{p = 2}$ and $\underline{q = 3}$. ①

(b) Hence, using algebraic integration, show that

$$\int_5^{10} \frac{3\,dx}{(x-1)\left(3+2\sqrt{x-1}\right)} = \ln a$$

where a is a rational constant to be found.

(6)

b) From part a : $\int_5^{10} \frac{3}{(x-1)(3+2\sqrt{x-1})}\,dx = \int_2^3 \frac{6}{u(3+2u)}\,du$

Partial Fractions : $\frac{6}{u(3+2u)} = \frac{A}{u} + \frac{B}{3+2u}$

$\Rightarrow \quad 6 = A(3+2u) + Bu$

let $u = 0 \Rightarrow 6 = 3A \Rightarrow A = 2$ and let $u = 1 \Rightarrow 6 = 10 + B$

$\Rightarrow B = -4$ 1

$\Rightarrow \int_2^3 \frac{6}{u(3+2u)} = \int_2^3 \frac{2}{u} - \frac{4}{3+2u}$ 1 $du = \left[2\ln(u) - 2\ln(3+2u)\right]_2^3$ 2

$-4\int \frac{1}{3+2u}\,du = \frac{-4\cdot\ln(3+2u)}{2}$

$= -2\ln(3+2u)$

$= \left[2\ln(3) - 2\ln(9)\right] - \left[2\ln(2) - 2\ln(7)\right]$

$\ln(a^b) = b\ln(a)$

$2\ln(9) = 2\ln(3^2) = 4\ln(3)$

$\log(a+b) = \log(ab)$

$\log(a-b) = \log\left(\frac{a}{b}\right)$

$= -2\ln(3) - 2\ln(2) + 2\ln(7)$

$= 2\ln(7/3) - 2\ln(2)$ 1

$= 2\ln\left(\frac{7}{3\times2}\right) = 2\ln\left(\frac{7}{6}\right) = \ln\left(\frac{7^2}{6^2}\right) = \ln\left(\frac{49}{36}\right) = \ln(a)$ 1

where $a = \frac{49}{36}$

3. (a) Use the substitution $u = 1 + \sqrt{x}$ to show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = \int_p^q \frac{2(u-1)^3}{u}\,du$$

where p and q are constants to be found.

(3)

(b) Hence show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = A - B\ln 5$$

where A and B are constants to be found.

(4)

a) $u = 1 + \sqrt{x}$

$u - 1 = \sqrt{x}$

$(u-1)^2 = x$ ①

use the shortened product rule for brackets:

$\frac{d}{dx}(ax+b)^n = a(n-1)(ax+b)^{n-1}$

$\frac{dx}{du} = 2(u-1)$

$\times du$

$\therefore dx = 2(u-1)\,du$

$$\int \frac{x}{1+\sqrt{x}}\,dx = \int \frac{(u-1)^2}{u} \times 2(u-1)\,du = \int \frac{2(u-1)^3}{u}\,du \quad ①$$

$x = 16, \quad u = 1 + \sqrt{16}, \quad u = 5$

$x = 0, \quad u = 1 + \sqrt{0}, \quad u = 1$

These are the new limits

$$\therefore \int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = \int_1^5 \frac{2(u-1)^3}{u}\,du \quad ①$$

where $p = 1$

$q = 5$

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expand brackets

b) $\int_0^5 \frac{2(u-1)^3}{u}\,du = 2\int_0^5 \frac{u^3-3u^2+3u-1}{u}\,du$

$\int \frac{1}{u}\,du = \ln u$

$= 2\int_0^5 \left(u^2-3u+3-\frac{1}{u}\right)du$ (1)

$= 2\left[\frac{u^3}{3}-\frac{3u^2}{2}+3u-\ln u\right]_0^5$ (1)

$= 2\left[\frac{5^3}{3}-\frac{3(5^2)}{2}+3(5)-\ln 5 - \left(\frac{1}{3}-\frac{3}{2}+3-\ln 1\right)\right]$ (1)

$= \frac{104}{3} - 2\ln 5$ (1)

4. (a) Use the substitution $u = 4 - \sqrt{h}$ to show that

$$\int \frac{dh}{4-\sqrt{h}} = -8\ln\left|4-\sqrt{h}\right| - 2\sqrt{h} + k$$

where k is a constant

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{dh}{dt} = \frac{t^{0.25}\left(4-\sqrt{h}\right)}{20}$$

where h is the height in metres and t is the time, measured in years, after the tree is planted.

(b) Find, according to the model, the range in heights of trees in this species.

(2)

One of these trees is one metre high when it is first planted.

According to the model,

(c) calculate the time this tree would take to reach a height of 12 metres, giving your answer to 3 significant figures.

(7)

a) $\int \frac{1}{4-\sqrt{h}}\, dh$

$u = 4 - \sqrt{h} \Rightarrow \sqrt{h} = 4 - u$

$du = -\frac{1}{2\sqrt{h}}\, dh \Rightarrow dh = -2\sqrt{h}\, du$ (1) $\Rightarrow dh = -2(4-u)\, du \Rightarrow \boxed{dh = -8+2u\, du}$

$\Rightarrow \int \frac{1}{4-\sqrt{h}}\, dh = \int \frac{1}{4-(4-u)}(-8+2u)\, du = \int \frac{2u-8}{u}\, du$ (1) $= \int \frac{2u}{u} - \frac{8}{u}\, du$

$= \int 2 - \frac{8}{u}\, du$ (1) $= 2u - 8\ln u + c$ (2)

$= 2(4-\sqrt{h}) - 8\ln\left|4-\sqrt{h}\right| + c$

$\Rightarrow 8 - 2\sqrt{h} - 8\ln\left|4-\sqrt{h}\right| + c$ $\qquad c + 8 = k$

$\Rightarrow \int \frac{1}{4-\sqrt{h}}\, dh = -8\ln\left|4-\sqrt{h}\right| - 2\sqrt{h} + k$ as required (1)

b) $\frac{dh}{dt} = \frac{t^{0.25}(4-\sqrt{h})}{20}$

• When the trees stop growing, $\frac{dh}{dt}$ (rate of change of height) is going to be equal to zero.

$\frac{dh}{dt} = \frac{t^{0.25}(4-\sqrt{h})}{20} = 0 \Rightarrow t^{0.25}(4-\sqrt{h}) = 0$

$\Rightarrow 4-\sqrt{h} = 0$ (1)

$\Rightarrow \sqrt{h} = 4$

$\Rightarrow h = 16m$

$\Rightarrow$ Range of heights of trees will be $\underline{0 \leq h \leq 16m}$ (1)

c) $\frac{dh}{dt} = \frac{t^{0.25}(4-\sqrt{h})}{20}$ and $t=0, h=1m \Rightarrow t=?$ when $h=12m$

$\int \frac{1}{4-\sqrt{h}}\, dh = \int \frac{t^{0.25}}{20}\, dt$ (1)

Part a

$\int \frac{t^{0.25}}{20}\, dt = \frac{t^{1.25}}{1.25 \times 20} = \frac{t^{1.25}}{25} + C$

$\Rightarrow -8\ln|4-\sqrt{h}| - 2\sqrt{h} = \frac{t^{1.25}}{25} \; (+c)$ (2)

Substitute back in!

$\Rightarrow t=0$ and $h=1 \Rightarrow -8\ln|4-\sqrt{1}| - 2\sqrt{1} = C$ (1)

$\Rightarrow -8\ln(3) - 2 = c$

$\Rightarrow -8\ln|4-\sqrt{h}| - 2\sqrt{h} = \frac{t^{1.25}}{25} - 8\ln(3) - 2$ (1)

$\Rightarrow h = 12m, t=? \Rightarrow -8\ln|4-\sqrt{12}| - 2\sqrt{12} = \frac{t^{1.25}}{25} - 8\ln(3) - 2$

$\Rightarrow \sqrt[1.25]{\left(-8\ln|4-\sqrt{12}| - 2\sqrt{12} + 8\ln(3) + 2\right)25} = t^{\cancel{1.25}}$ (1)

$\Rightarrow t = \underline{75.2}$ Years when $h = 12m$ (1)