1.

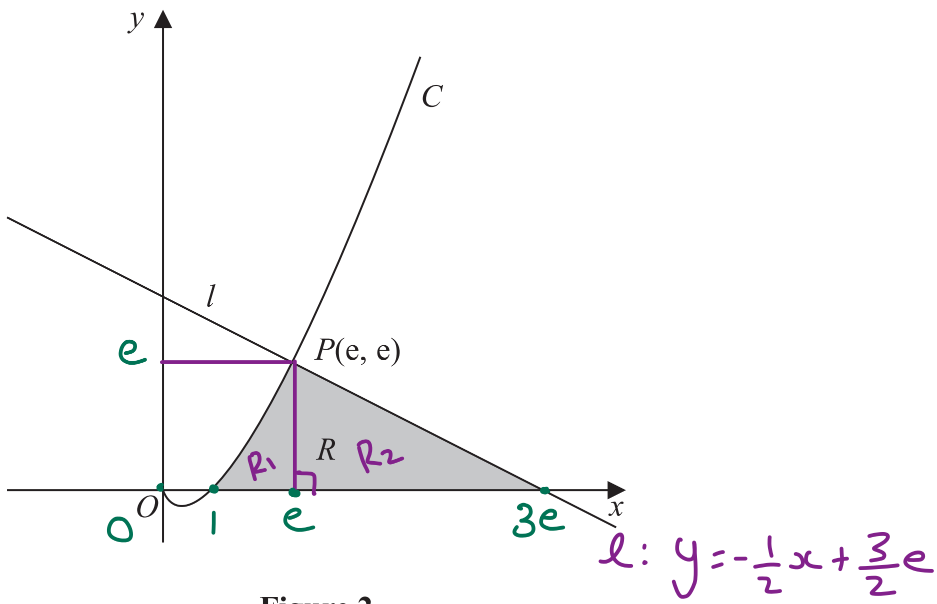

Figure 2

Figure 2 shows a sketch of part of the curve C with equation $y = x\ln x$, $\quad x > 0$

The line l is the normal to C at the point $P(\mathrm{e}, \mathrm{e})$

The region R, shown shaded in Figure 2, is bounded by the curve C, the line l and the x-axis.

Show that the exact area of R is $A\mathrm{e}^2 + B$ where A and B are rational numbers to be found.

(10)

finding equation of line l:

Since line l is a normal to C at P, $m_l = -\dfrac{1}{m_t}$

m_l = gradient of l at P

m_t = gradient of tangent to C at P.

$m_t = \left.\dfrac{dy}{dx}\right|_{x=e}$ $\qquad$ $\dfrac{dy}{dx} = \dfrac{d}{dx}(x\ln x)$

$= 1 + \ln e$ $\qquad$ $= x\dfrac{d}{dx}(\ln x) + \ln x \dfrac{d}{dx}(x)$

$= 1 + 1$

$m_t = 2$ $\qquad$ $= 1 + \ln x$

$m_l = -\dfrac{1}{m_t} = -\dfrac{1}{2}$

Using $y - y_1 = m_l(x - x_1)$ to find l:

$y_1 = e$, $x_1 = e$ (point P) $\quad \Rightarrow y - e = -\dfrac{1}{2}(x - e)$

$$y = e - \frac{1}{2}x + \frac{1}{2}e$$

$$\therefore y = -\frac{1}{2}x + \frac{3}{2}e$$

finding where l intersects x-axis:

$\rightarrow$ at x-axis, $y = 0$

$$-\frac{1}{2}x + \frac{3}{2}e = 0$$

$$\frac{1}{2}x = \frac{3}{2}e$$

$$x = 3e$$

finding where C intersects x-axis:

$\rightarrow$ at x-axis, $y = 0$

$$x\ln x = 0$$

$$\rightarrow x = 0$$

$$\rightarrow \ln x = 0 \Rightarrow x = 1$$

$$\text{area } R_1 = \int_1^e x\ln x \, dx$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$u = \ln x \quad \frac{dv}{dx} = x$$

$$\frac{du}{dx} = \frac{1}{x} \quad v = \frac{x^2}{2}$$

$$= \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2} \times \frac{1}{x}\,dx$$

$$= \left[\frac{x^2}{2}\ln x\right]_1^e - \left[\frac{x^2}{4}\right]_1^e$$

$$= \left(\frac{e^2}{2}\overbrace{\ln e}^{1} - \underbrace{\frac{1^2}{2}\overbrace{\ln 1}^{0}}_{0}\right) - \left(\frac{e^2}{4} - \frac{1^2}{4}\right)$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2}{4} + \frac{1}{4}$$

$$\text{area } R_2 = \frac{1}{2} \times (3e - e) \times e$$

$$= \frac{1}{2}(2e)(e)$$

$$= e^2$$

$$\text{Area } R = R_1 + R_2$$

$$= \frac{e^2}{4} + \frac{1}{4} + e^2$$

$$= \frac{e^2}{4} + \frac{1}{4} + \frac{4e^2}{4} = \frac{5e^2}{4} + \frac{1}{4}$$

$$A = \frac{5}{4}, \ B = \frac{1}{4}$$

2.

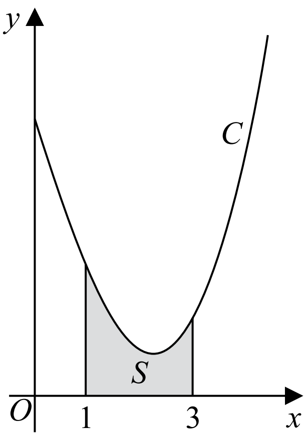

Figure 4

Figure 4 shows a sketch of part of the curve C with equation

$$y = \frac{x^2 \ln x}{3} - 2x + 5, \quad x > 0$$

The finite region S, shown shaded in Figure 4, is bounded by the curve C, the line with equation $x = 1$, the x-axis and the line with equation $x = 3$

The table below shows corresponding values of x and y with the values of y given to 4 decimal places as appropriate.

$\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

x	1	1.5	2	2.5	3
y	3	2.3041	1.9242	1.9089	2.2958

y_0 y_1 y_2 y_3 y_4

(a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of S, giving your answer to 3 decimal places.

(3)

(b) Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of S.

(1)

(c) Show that the exact area of S can be written in the form $\frac{a}{b} + \ln c$, where a, b and c are integers to be found.

(6)

(In part c, solutions based entirely on graphical or numerical methods are not acceptable.)

a) $A = \frac{1}{2} \times h\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\right]$ Trapezium Rule

$h = 1.5 - 1 = 2 - 1.5 = 0.5 \Rightarrow h = 0.5$ (1)

$A = \frac{1}{2} \times \frac{1}{2}\left[(3 + 2.2958) + 2(2.3041 + 1.9242 + 1.9089)\right] = 4.39255$

$\Rightarrow$ Area of S is $\underline{4.393}$ (3 d.p) (1)

Question continued

b) • h is the width of intervals

$\Rightarrow$ Option 1: decrease h (width of the strips) 1

Option 2: increase the number of strips

c)

$y = \dfrac{x^2 \ln x}{3} - 2x + 5$

$A = \displaystyle\int_1^3 \frac{x^2 \ln x}{3} - 2x + 5 \, dx$ Integration by parts: 1

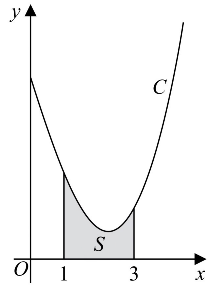

Figure 4

$\displaystyle\int \frac{x^2 \ln x}{3} \, dx = f(x) \cdot g(x) - \int f(x) \cdot g'(x) \, dx$

let $f'(x) = \dfrac{x^2}{3}$ $f(x) = \dfrac{x^3}{9}$ $\Rightarrow \displaystyle\int \frac{x^2 \ln x}{3} \, dx = \frac{x^3}{9}\ln(x) - \int \frac{x^3}{9} \cdot \frac{1}{x} \, dx$ 1

$g(x) = \ln(x)$ $g'(x) = \dfrac{1}{x}$ $= \dfrac{x^3}{9}\ln(x) - \dfrac{1}{9}\displaystyle\int x^2 \, dx$

$= \dfrac{x^3}{9}\ln(x) - \dfrac{x^3}{27} + C$ 1

$\Rightarrow A = \displaystyle\int_1^3 \frac{x^2 \ln x}{3} - 2x + 5 \, dx$

$\Rightarrow A = \left[\dfrac{x^3}{9}\ln x - \dfrac{x^3}{27} - x^2 + 5x\right]_1^3 = \left[\dfrac{3^3}{9}\ln(3) - \underbrace{\dfrac{3^3}{27} - 9 + 15}_{-1-9+15}\right] - \left[\underbrace{\dfrac{1}{9}\ln(1)}_{=0} \underbrace{- \dfrac{1}{27} - 1 + 5}_{\frac{107}{27}}\right]$ 1

$\ln(1) = 0$

$\Rightarrow A = (3\ln(3) + 5) - (107/27)$

$*\ a\ln(b) = \ln(b^a)$

$\Rightarrow A = 3\ln(3) + \dfrac{28}{27}$

$\Rightarrow A = \ln(27) + \dfrac{28}{27}$ $a = 28$, $b = 27$ and $c = 27$. 1