1. The curve C with equation

$$y = \frac{p - 3x}{(2x - q)(x + 3)} \qquad x \in \mathbb{R}, x \neq -3, x \neq 2$$

where p and q are constants, passes through the point $\left(3, \frac{1}{2}\right)$ and has two vertical asymptotes with equations $x = 2$ and $x = -3$

(a) (i) Explain why you can deduce that $q = 4$

(ii) Show that $p = 15$

(3)

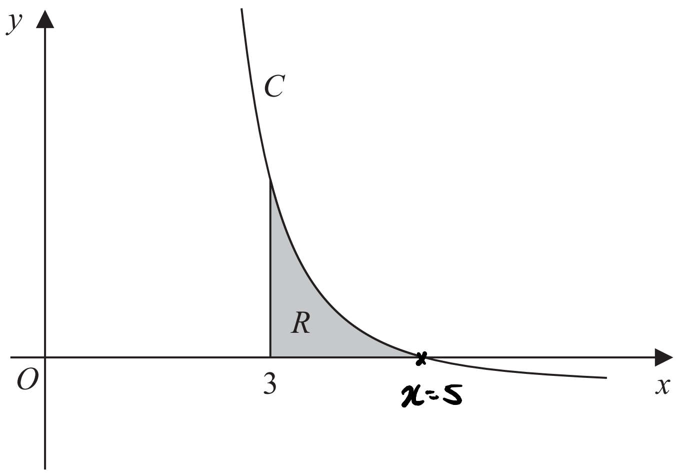

Figure 4

Figure 4 shows a sketch of part of the curve C. The region R, shown shaded in Figure 4, is bounded by the curve C, the x-axis and the line with equation $x = 3$

(b) Show that the exact value of the area of R is $a \ln 2 + b \ln 3$, where a and b are rational constants to be found.

(8)

a i) Verticle asymtotes when $(2x-q)(x+3)=0$

$2x-q=0 \qquad x+3=0$

$2x=q \qquad x=-3$

① $x=2$

$2(2)=q$

$\therefore q=4$ as needed

a ii) $y=\dfrac{p-3x}{(2x-4)(x+3)}$

$\frac{1}{2}=\dfrac{p-3(3)}{(2(3)-4)(3+3)}=\dfrac{p-9}{12}$

$2 \times 6 = 12$ ① $(3, \frac{1}{2})$

$\frac{1}{2}=\dfrac{p-9}{12}$

$6=p-9$ ①

$p=6+9$

$\therefore p=15$ as needed

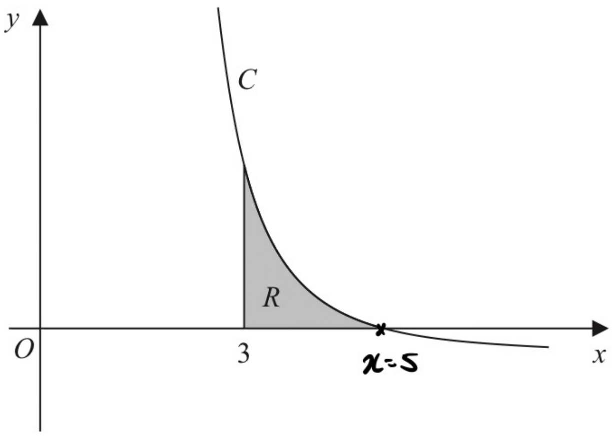

Figure 4

Figure 4 shows a sketch of part of the curve C. The region R, shown shaded in Figure 4, is bounded by the curve C, the x-axis and the line with equation $x = 3$

(b) Show that the exact value of the area of R is $a\ln 2 + b\ln 3$, where a and b are rational constants to be found.

(8)

$$\frac{15-3x}{(2x-4)(x+3)} = \frac{1.8}{(2x-4)} - \frac{2.4}{(x+3)}$$ 1

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int_3^5 \frac{1.8}{(2x-4)} - \frac{2.4}{(x+3)}\,dx = \Big[0.9\ln|2x-4| - 2.4\ln|x+3|\Big]_3^5$$

$f(x) = 2x-4$
$f'(x) = 2$
$\frac{1.8}{2} = 0.9$

$f(x) = x+3$
$f'(x) = 1$
$\frac{2.4}{1} = 2.4$

4

$= 0.9\ln|2(5)-4| - 2.4\ln|5+3| - \big[0.9\ln|2(3)-4| - 2.4\ln|3+3|\big]$

$= 0.9\ln|6| - 2.4\ln|8| - 0.9\ln|2| + 2.4\ln|6|$

using log laws
$\ln(a\times b) = \ln a + \ln b$
$\ln a^k = k\ln a$

$= 3.3\ln|6| - 2.4\ln|8| - 0.9\ln|2|$

$= 3.3(\ln|3| + \ln|2|) - 2.4\ln|2^3| - 0.9\ln|2|$

$= 3.3\ln|3| + 3.3\ln|2| - 7.2\ln|2| - 0.9\ln|2|$

$= 3.3\ln|3| - 4.8\ln|2|$ 1

2.

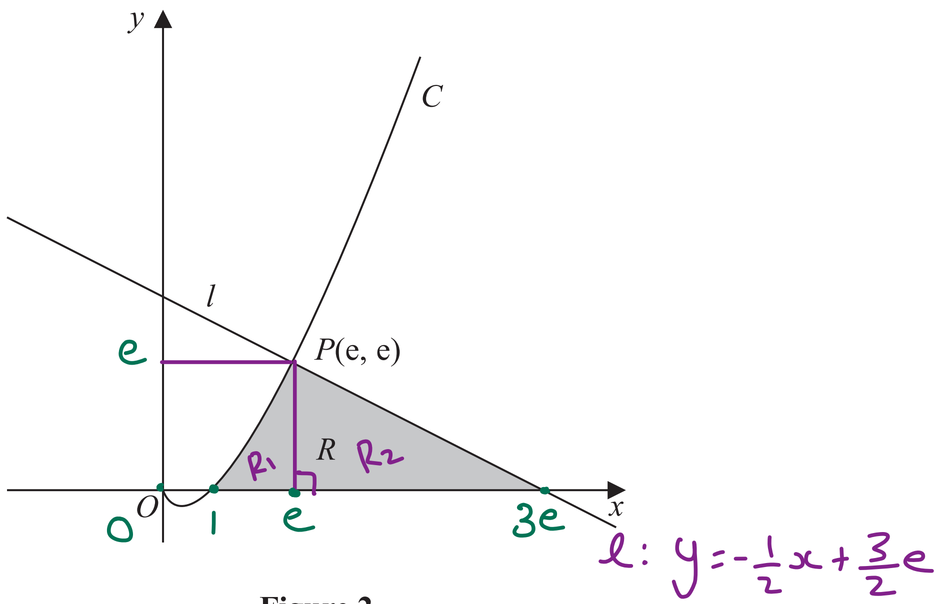

Figure 2

Figure 2 shows a sketch of part of the curve C with equation $y = x\ln x, \quad x > 0$

The line l is the normal to C at the point $P(\mathrm{e}, \mathrm{e})$

The region R, shown shaded in Figure 2, is bounded by the curve C, the line l and the x-axis.

Show that the exact area of R is $A\mathrm{e}^2 + B$ where A and B are rational numbers to be found.

(10)

finding equation of line l:

Since line l is a normal to C at P, $m_l = -\frac{1}{m_t}$

m_l = gradient of l at P

m_t = gradient of tangent to C at P.

$m_t = \left.\frac{dy}{dx}\right|_{x=e}$ $\qquad \frac{dy}{dx} = \frac{d}{dx}(x\ln x)$

$= 1 + \ln e$ $\qquad = x\frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(x)$

$= 1 + 1$

$m_t = 2$ $\qquad = 1 + \ln x$

$m_l = -\frac{1}{m_t} = -\frac{1}{2}$

Using $y - y_1 = m_l(x - x_1)$ to find l:

$y_1 = e$, $x_1 = e$ (point P) $\Rightarrow y - e = -\frac{1}{2}(x - e)$

$$y = e - \frac{1}{2}x + \frac{1}{2}e$$

$$\therefore y = -\frac{1}{2}x + \frac{3}{2}e$$

finding where l intersects x-axis:
→ at x-axis, $y = 0$

$$-\frac{1}{2}x + \frac{3}{2}e = 0$$

$$\frac{1}{2}x = \frac{3}{2}e$$

$$x = 3e$$

finding where C intersects x-axis:
→ at x-axis, $y = 0$

$$x \ln x = 0$$

$$\rightarrow x = 0$$

$$\rightarrow \ln x = 0 \Rightarrow x = 1$$

$$\text{area } R_1 = \int_1^e x \ln x \, dx$$

$$\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$$

$$u = \ln x \quad \frac{dv}{dx} = x$$

$$\frac{du}{dx} = \frac{1}{x} \quad v = \frac{x^2}{2}$$

$$= \left[\frac{x^2}{2} \ln x\right]_1^e - \int_1^e \frac{x^2}{2} \times \frac{1}{x} \, dx$$

$$= \left[\frac{x^2}{2} \ln x\right]_1^e - \left[\frac{x^2}{4}\right]_1^e$$

$$= \left(\frac{e^2}{2} \overbrace{\ln e}^{1} - \underbrace{\frac{1^2}{2} \overbrace{\ln 1}^{0}}_{0}\right) - \left(\frac{e^2}{4} - \frac{1^2}{4}\right)$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2}{4} + \frac{1}{4}$$

area $R_2 = \frac{1}{2} \times (3e - e) \times e$

$= \frac{1}{2}(2e)(e)$

$= e^2$

Area $R = R_1 + R_2$

$= \frac{e^2}{4} + \frac{1}{4} + e^2$

$= \frac{e^2}{4} + \frac{1}{4} + \frac{4e^2}{4} = \frac{5e^2}{4} + \frac{1}{4}$

$A = \frac{5}{4}, B = \frac{1}{4}$

3.

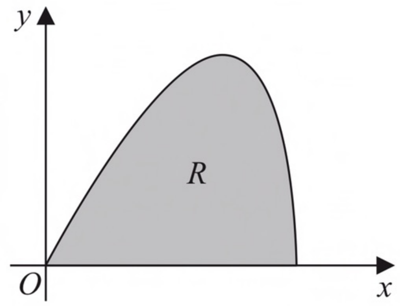

Figure 3

The curve shown in Figure 3 has parametric equations

$$x = 6\sin t \qquad y = 5\sin 2t \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

The region R, shown shaded in Figure 3, is bounded by the curve and the x-axis.

(a) (i) Show that the area of R is given by $\displaystyle\int_0^{\frac{\pi}{2}} 60\sin t\cos^2 t\,\mathrm{d}t$

(3)

$$R = \int_{x_1}^{x_2} y(x)\ dx = \int_{t_1}^{t_2} y(t)\ \frac{dx}{dt}\ dt$$

$y(t) = y = 5\sin(2t)$

$x = 6\sin t \Rightarrow \frac{dx}{dt} = 6\cos t$

$$= \int_{t_1}^{t_2} 6\cos t \cdot 5\sin 2t\ dt \quad 1$$

• $\sin 2t = 2\sin t\cos t$

$$= \int_{t_1}^{t_2} 30\cos t \cdot \sin 2t$$

$$= \int_{t_1}^{t_2} 30\cos t \cdot 2\sin t\cos t\ dt = \int_{t_1}^{t_2} 60\cos^2 t\sin t\ dt \quad 1$$

$\Rightarrow t_1 = 0$ and $t_2 = \frac{\pi}{2} \Rightarrow R = \displaystyle\int_0^{\pi/2} 60\sin t\cos^2 t\ dt$ as required. 1

(ii) Hence show, by algebraic integration, that the area of R is exactly 20

(3)

From part i: $R = \int_0^{\pi/2} 60 \cdot \sin t \cdot \cos^2 t \, dt$

$\cos^2 t = 1 - \sin^2 t$

$= 60 \cdot \int_0^{\pi/2} \sin t (1 - \sin^2 t) \, dt$

$\int \sin t \, dt = -\cos t + C$

$\int -\sin^3 t \, dt = \cos t - \frac{1}{3}\cos^3(t) + C$

$= 60 \int_0^{\pi/2} \sin t - \sin^3 t \, dt$ (1)

$= 60 \left[-\frac{1}{3}\cos^3 t \right]_0^{\pi/2}$ (1) $= 60 \left[-\frac{1}{3}\cos^3\left(\frac{\pi}{2}\right) - -\frac{1}{3}\cos^3(0) \right]$

$\Rightarrow R = 60\left(0 - -\frac{1}{3}\right)$

$\Rightarrow R = \underline{20}$ as required. (1)

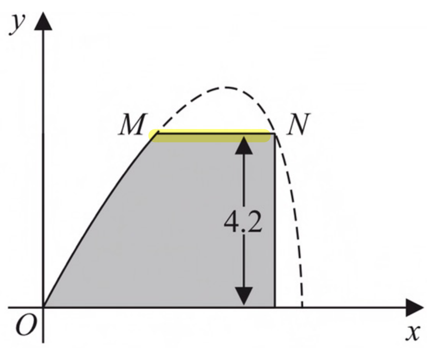

Figure 4

Part of the curve is used to model the profile of a small dam, shown shaded in Figure 4.
Using the model and given that

- x and y are in metres
- the vertical wall of the dam is 4.2 metres high
- there is a horizontal walkway of width MN along the top of the dam

S | A
T | C

(b) calculate the width of the walkway.

(5)

$x = 6\sin t$, $y = 5\sin 2t \Rightarrow$ when $y = 4.2 \Rightarrow 4.2 = 5\sin 2t$

$\Rightarrow \sin 2t = \frac{4.2}{5}$ (1) $\Rightarrow 2t = \sin^{-1}\left(\frac{4.2}{5}\right) \Rightarrow t_1 = \underline{0.49865...}$

and $t_2 = \pi - 2 \times t_1 = \pi - 2 \times 0.49865 = \underline{1.0721}...$ (1)

$\Rightarrow x_1 = 6\sin(0.49865) = \underline{2.869}$ and $x_2 = 6\sin(1.0721) = \underline{5.269}$ (1) (1)

Width of the path is going to be $x_2 - x_1 = 5.269 - 2.869$

$= \underline{2.40\text{m}}$ (1)

4.

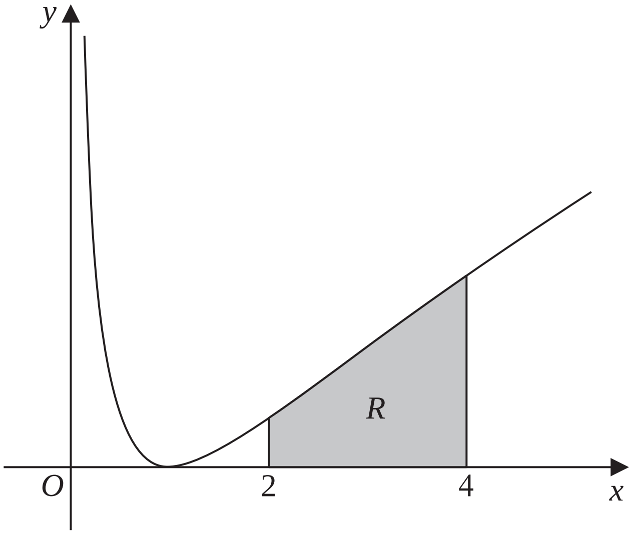

Figure 2

Figure 2 shows a sketch of part of the curve with equation

$$y = (\ln x)^2 \qquad x > 0$$

The finite region R, shown shaded in Figure 2, is bounded by the curve, the line with equation $x = 2$, the x-axis and the line with equation $x = 4$

The table below shows corresponding values of x and y, with the values of y given to 4 decimal places.

$+0.5$ $\quad +0.5$ $\quad +0.5$ $\quad +0.5$ $\quad \therefore h = 0.5$

x	2	2.5	3	3.5	4
y	0.4805	0.8396	1.2069	1.5694	1.9218

$y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4$

(a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of R, giving your answer to 3 significant figures.

(3)

(b) Use algebraic integration to find the exact area of R, giving your answer in the form

$$y = a(\ln 2)^2 + b\ln 2 + c$$

where a, b and c are integers to be found.

(5)

(a) $h = 0.5$ 1

$$\int_2^4 (\ln x)^2\,dx \approx \frac{1}{2} \times 0.5 \times \{0.4805 + 1.9218 + 2(0.8396 + 1.2069 + 1.5694)\}$$ 1

$= 2.408525$

Area of R $= 2.41$ units^2 (3sf) 1

DO NOT WRITE IN THIS AREA

Question continued

(b) $\int_2^4 (\ln x)^2 \, dx \equiv \int_2^4 \ln x \times \ln x \, dx$ — u, $\frac{dv}{dx}$

$u = \ln x \qquad v = ?$

$u' = \frac{1}{x} \qquad v' = \ln x$

Formula:

$$\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$$

To find v, we integrate v':

$\int \ln x \, dx \equiv \int \ln x \times 1 \, dx$ — u, v'

$u = \ln x \qquad v = x$

$u' = \frac{1}{x} \qquad v' = 1$

$\therefore \int \ln x \, dx = x \ln x - \int x \left(\frac{1}{x}\right) dx$

$= x \ln x - x$

$\int_2^4 (\ln x)^2 \, dx \equiv \int_2^4 \ln x \times \ln x \, dx$

$u = \ln x \qquad v = x \ln x - x$

$u' = \frac{1}{x} \qquad v' = \ln x$

$\therefore \int_2^4 (\ln x)^2 \, dx = \ln x \,(x \ln x - x) - \int (x \ln x - x)\left(\frac{1}{x}\right) dx$ (1)

$= x \ln x (\ln x - 1) - \int (\ln x - 1) \, dx$

$= x \ln x (\ln x - 1) - \{ x \ln x - x - x \} + c$

$= x(\ln x)^2 - x \ln x - x \ln x + 2x + c$ (1)

$= x(\ln x)^2 - 2x \ln x + 2x + c$ (1)

Question continued

$$\therefore \int_2^4 (\ln x)^2\,dx = \left[x(\ln x)^2 - 2x\ln x + 2x\right]_2^4$$

$$= \{4(\ln 4)^2 - 2(4)\ln 4 + 2(4)\} - \{2(\ln 2)^2 - 2(2)\ln 2 + 2(2)\}$$

$$= \{4(\ln 2^2)^2 - 8\ln 2^2 + 8\} - \{2(\ln 2)^2 - 4\ln 2 + 4\}$$

$$= \{4(2\ln 2)^2 - 16\ln 2 + 8\} - \{2(\ln 2)^2 - 4\ln 2 + 4\}$$

$$= 16(\ln 2)^2 - 16\ln 2 + 8 - 2(\ln 2)^2 + 4\ln 2 - 4 \quad 1$$

$$= 14(\ln 2)^2 - 12\ln 2 + 4 \quad 1$$

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA