1. Given that a is a positive constant and

$$\int_a^{2a} \frac{t+1}{t}\,\mathrm{d}t = \ln 7$$

show that $a = \ln k$, where k is a constant to be found.

(4)

$$\int_a^{2a} \frac{t+1}{t}\,dt = \int_a^{2a} \frac{t}{t} + \frac{1}{t}\,dt = \int_a^{2a} 1 + \frac{1}{t}\,dt = \left[t + \ln(t)\right]_a^{2a} \quad 1$$

$$\Rightarrow \left(2a + \ln(2a)\right) - \left(a + \ln(a)\right) = \ln(7) \quad 1$$

$$\Rightarrow 2a + \ln(2) + \ln(a) - a - \ln(a) = \ln(7)$$

$$\Rightarrow a + \ln(2) = \ln(7)$$

$\log(ab) = \log(a) + \log(b)$

$\log(a/b) = \log(a) - \log(b)$

$$\Rightarrow a = \ln(7) - \ln(2) = \ln(7/2)$$

$$\Rightarrow a = \ln\left(\frac{7}{2}\right) = \ln(k) \quad \text{where } \underline{\underline{k = \frac{7}{2}}}.$$

2. Given that $k \in \mathbb{Z}^+$

(a) show that $\int_k^{3k} \frac{2}{(3x-k)} \mathrm{d}x$ is independent of k,

↳ No k term/variable.

(4)

(b) show that $\int_k^{2k} \frac{2}{(2x-k)^2} \mathrm{d}x$ is inversely proportional to k.

↳ of form $\frac{A}{k}$ (i.e. $\propto 1/k$)

(3)

a) $\int \frac{a}{(bx+c)} dx = \frac{a}{b} \ln|bx+c| + C$

$$\int \frac{2}{(3x-k)} dx = \frac{2}{3} \ln(3x-k)$$

$$\int_k^{3k} \frac{2}{(3x-k)} dx = \left[\frac{2}{3}\ln(3x-k)\right]_k^{3k}$$

$$= \frac{2}{3}\ln(9k-k) - \frac{2}{3}\ln(3k-k)$$

$$= \frac{2}{3}\left[\ln(8k) - \ln(2k)\right]$$

$\ln A - \ln B = \ln\left(\frac{A}{B}\right)$

$$= \frac{2}{3} \times \ln\left(\frac{8k}{2k}\right) = \frac{2}{3}\ln(4)$$

Our result for the integral has not got the term k present, ∴ it is independent of k, b)

$$\int \frac{\cancel{2}}{u^2} \times \frac{1}{\cancel{2}}\, du \qquad u = 2x - k$$

$$\frac{du}{dx} = 2 \Rightarrow dx = \frac{1}{2}\, du$$

$$\int \frac{1}{u^2}\, du = \int u^{-2}\, du = -u^{-1} \text{ or } -\frac{1}{u}$$

$$\therefore \int \frac{2}{(2x-k)^2}\, dx = -\frac{1}{(2x-k)}$$

$$\Rightarrow \int_k^{2k} \frac{2}{(2x-k)^2}\, dx = \left[-\frac{1}{2x-k}\right]_k^{2k}$$

$$= -\frac{1}{4k-k} - -\frac{1}{2k-k}$$

$$= -\frac{1}{3k} + \frac{1}{k}$$

$$= \frac{3}{3k} - \frac{1}{3k} = \frac{2}{3k}$$

the integral result is $\frac{2}{3k}$, which is of form $\frac{A}{bk}$, and $\therefore$ means $\propto \frac{1}{k}$

3.

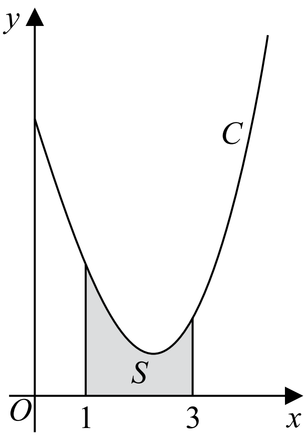

Figure 4

Figure 4 shows a sketch of part of the curve C with equation

$$y = \frac{x^2 \ln x}{3} - 2x + 5, \quad x > 0$$

The finite region S, shown shaded in Figure 4, is bounded by the curve C, the line with equation $x = 1$, the x-axis and the line with equation $x = 3$

The table below shows corresponding values of x and y with the values of y given to 4 decimal places as appropriate.

x	1	1.5	2	2.5	3
y	3	2.3041	1.9242	1.9089	2.2958
	y_0	y_1	y_2	y_3	y_4

(a) Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of S, giving your answer to 3 decimal places.

(3)

(b) Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of S.

(1)

(c) Show that the exact area of S can be written in the form $\frac{a}{b} + \ln c$, where a, b and c are integers to be found.

(6)

(In part c, solutions based entirely on graphical or numerical methods are not acceptable.)

a) $A = \frac{1}{2} \times h\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})\right]$ Trapezium Rule

$h = 1.5 - 1 = 2 - 1.5 = 0.5 \Rightarrow h = 0.5$ 1

$A = \frac{1}{2} \times \frac{1}{2}\left[(3 + 2.2958) + 2(2.3041 + 1.9242 + 1.9089)\right] = 4.39255$

$\Rightarrow$ Area of S is $\underline{4.393}$ (3 d.p) 1

b) • h is the width of intervals

$\Rightarrow$ Option 1: decrease h (width of the strips) 1

Option 2: increase the number of strips

c)

$y = \frac{x^2 \ln x}{3} - 2x + 5$

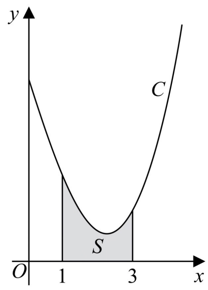

Figure 4

$A = \int_1^3 \frac{x^2 \ln x}{3} - 2x + 5 \; dx$ Integration by parts: 1

$\int \frac{x^2 \ln x}{3} \, dx = f(x) \cdot g(x) - \int f(x) \cdot g'(x) \, dx$

let $f'(x) = \frac{x^2}{3}$ $\quad f(x) = \frac{x^3}{9}$ $\Rightarrow \int \frac{x^2 \ln x}{3} \, dx = \frac{x^3}{9} \ln(x) - \int \frac{x^3}{9} \cdot \frac{1}{x} \, dx$ 1

$g(x) = \ln(x)$ $\quad g'(x) = \frac{1}{x}$ $\quad = \frac{x^3}{9} \ln(x) - \frac{1}{9} \int x^2 \, dx$

$= \frac{x^3}{9} \ln(x) - \frac{x^3}{27} + C$ 1

$\Rightarrow A = \int_1^3 \frac{x^2 \ln x}{3} - 2x + 5 \; dx$

$\Rightarrow A = \left[\frac{x^3}{9} \ln x - \frac{x^3}{27} - x^2 + 5x \right]_1^3$ 1 $= \left[\frac{3^3}{9} \ln(3) - \frac{3^3}{27} - 9 + 15 \right] - \left[\underbrace{\frac{1}{9} \ln(1)}_{=0} - \underbrace{\frac{1}{27} - 1 + 5}_{\frac{107}{27}} \right]$

$-1 - 9 + 15$ $\quad \ln(1) = 0$

$\Rightarrow A = (3\ln(3) + 5) - (107/27)$

$*\ a \ln(b) = \ln(b^a)$

$\Rightarrow A = 3\ln(3) + \frac{28}{27}$

$\Rightarrow A = \ln(27) + \frac{28}{27}$ $\qquad a = 28,\ b = 27$ and $c = 27.$ 1

4. Show that

$$\int_0^2 2x\sqrt{x+2}\,\mathrm{d}x = \frac{32}{15}\left(2+\sqrt{2}\right)$$

(7)

$$\int_0^2 2x(x+2)^{1/2}\,dx$$

let $u = x+2$

$\frac{du}{dx} = 1$

$du = dx$

$$= \int_2^4 (2u-4)(u^{1/2})\,du$$

$x+2 = u$

$\therefore x = u-2$

@ $x = 2,\ u = 4$

@ $x = 0,\ u = 2$

$$= \int_2^4 2u^{3/2} - 4u^{1/2}\,du$$

$$= \left[\frac{2u^{5/2}}{5/2} - \frac{4u^{3/2}}{3/2}\right]_2^4$$

$$= \left[\frac{4u^{5/2}}{5} - \frac{8u^{3/2}}{3}\right]_2^4$$

$$= \left(\frac{4\times 4^{5/2}}{5} - \frac{8\times 4^{3/2}}{3}\right) - \left(\frac{4\times 2^{5/2}}{5} - \frac{8\times 2^{3/2}}{3}\right)$$

$$= \frac{128}{5} - \frac{64}{3} - \frac{16\sqrt{2}}{5} + \frac{16\sqrt{2}}{3}$$

$$= \frac{64+32\sqrt{2}}{15} = \frac{32(2+\sqrt{2})}{15}$$

$$\frac{ab}{c} = \frac{a}{c}\times b$$

$= \frac{32}{15}(2+\sqrt{2})$ as required.

5.

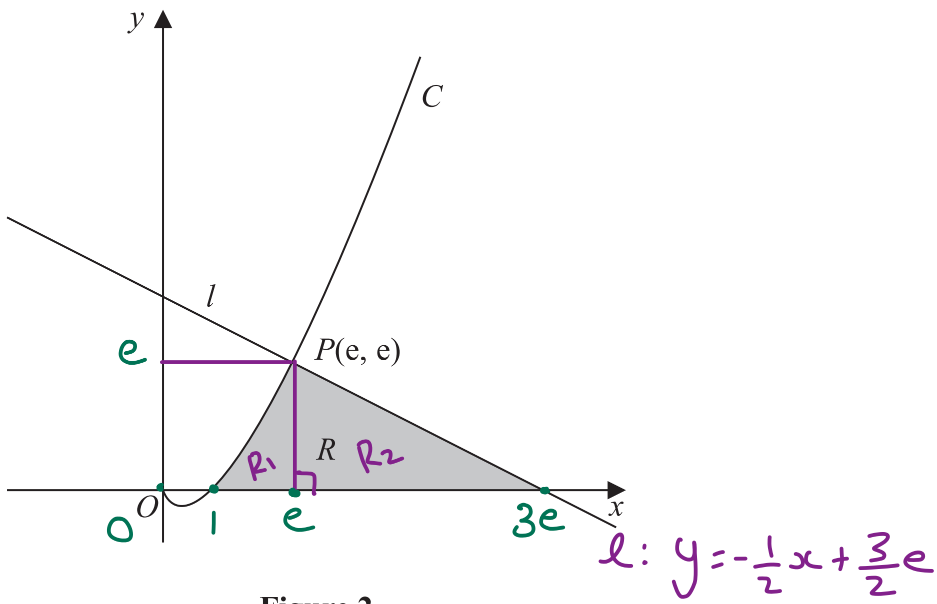

Figure 2

Figure 2 shows a sketch of part of the curve C with equation $y = x\ln x$, $\quad x > 0$

The line l is the normal to C at the point $P(\mathrm{e}, \mathrm{e})$

The region R, shown shaded in Figure 2, is bounded by the curve C, the line l and the x-axis.

Show that the exact area of R is $A\mathrm{e}^2 + B$ where A and B are rational numbers to be found.

(10)

finding equation of line l:

Since line l is a normal to C at P, $m_l = -\frac{1}{m_t}$

m_l = gradient of l at P

m_t = gradient of tangent to C at P.

$m_t = \left.\frac{dy}{dx}\right|_{x=e}$ $\qquad$ $\frac{dy}{dx} = \frac{d}{dx}(x\ln x)$

$= 1 + \ln e$ $\qquad$ $= x\frac{d}{dx}(\ln x) + \ln x \frac{d}{dx}(x)$

$= 1 + 1$

$m_t = 2$ $\qquad$ $= 1 + \ln x$

$m_l = -\frac{1}{m_t} = -\frac{1}{2}$

Using $y - y_1 = m_l(x - x_1)$ to find l:

$y_1 = e$, $x_1 = e$ (point P) $\Rightarrow y - e = -\frac{1}{2}(x - e)$

$$y = e - \frac{1}{2}x + \frac{1}{2}e$$

$$\therefore y = -\frac{1}{2}x + \frac{3}{2}e$$

finding where l intersects x-axis:
→ at x-axis, $y = 0$

$$-\frac{1}{2}x + \frac{3}{2}e = 0 \checkmark$$

$$\frac{1}{2}x = \frac{3}{2}e$$

$$x = 3e \checkmark$$

finding where C intersects x-axis:
→ at x-axis, $y = 0$

$$x \ln x = 0$$
$$\rightarrow x = 0$$
$$\rightarrow \ln x = 0 \Rightarrow x = 1$$

area $R_1 \checkmark = \int_1^e x \ln x \, dx$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$u = \ln x$ $\quad \frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x}$ $\quad v = \frac{x^2}{2}$

$$= \left[\frac{x^2}{2} \ln x\right]_1^e - \int_1^e \frac{x^2}{2} \times \frac{1}{x} \, dx \checkmark$$

$$= \left[\frac{x^2}{2} \ln x\right]_1^e - \left[\frac{x^2}{4}\right]_1^e \checkmark\checkmark$$

$$= \left(\frac{e^2}{2} \overbrace{\ln e}^{1} - \underbrace{\frac{1^2}{2} \overbrace{\ln 1}^{0}}_{0}\right) - \left(\frac{e^2}{4} - \frac{1^2}{4}\right)$$

$$= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{e^2}{4} + \frac{1}{4}$$

area $R_2 = \frac{1}{2} \times (3e - e) \times e$

$= \frac{1}{2}(2e)(e)$

$= e^2$

Area $R = R_1 + R_2$ ✓

$= \frac{e^2}{4} + \frac{1}{4} + e^2$

$= \frac{e^2}{4} + \frac{1}{4} + \frac{4e^2}{4} = \frac{5e^2}{4} + \frac{1}{4}$ ✓

$A = \frac{5}{4}, B = \frac{1}{4}$

6. The curve C with equation

$$y = \frac{p - 3x}{(2x - q)(x + 3)} \qquad x \in \mathbb{R}, x \neq -3, x \neq 2$$

where p and q are constants, passes through the point $\left(3, \frac{1}{2}\right)$ and has two vertical asymptotes with equations $x = 2$ and $x = -3$

(a) (i) Explain why you can deduce that $q = 4$

(ii) Show that $p = 15$

(3)

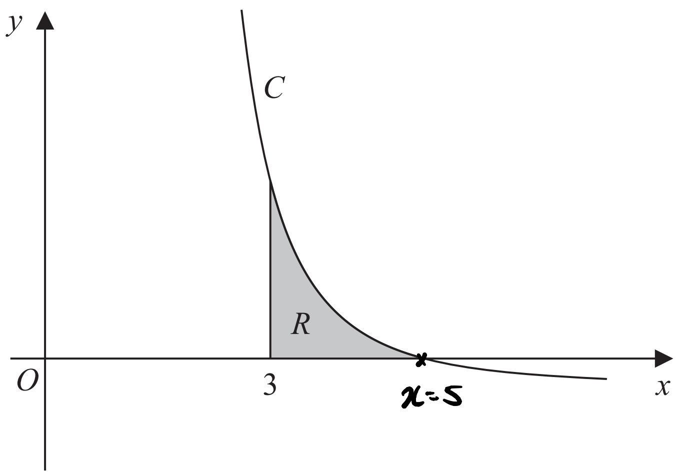

Figure 4

Figure 4 shows a sketch of part of the curve C. The region R, shown shaded in Figure 4, is bounded by the curve C, the x-axis and the line with equation $x = 3$

(b) Show that the exact value of the area of R is $a\ln 2 + b\ln 3$, where a and b are rational constants to be found.

(8)

a i) Verticle asymtotes when $(2x-q)(x+3)=0$

$2x-q=0$ $\quad$ $x+3=0$

$2x=q$ $\quad$ $x=-3$

1 $\quad x=2$

$2(2)=q$

$\therefore q=4$ as needed

a ii) $y=\dfrac{p-3x}{(2x-4)(x+3)}$

$\frac{1}{2}=\dfrac{p-3(3)}{(2(3)-4)(3+3)}=\dfrac{p-9}{12}$

$2 \times 6 = 12$ 1 $(3, \frac{1}{2})$

$\frac{1}{2}=\dfrac{p-9}{12}$

$6=p-9$ 1

$p=6+9$

$\therefore p=15$ as needed

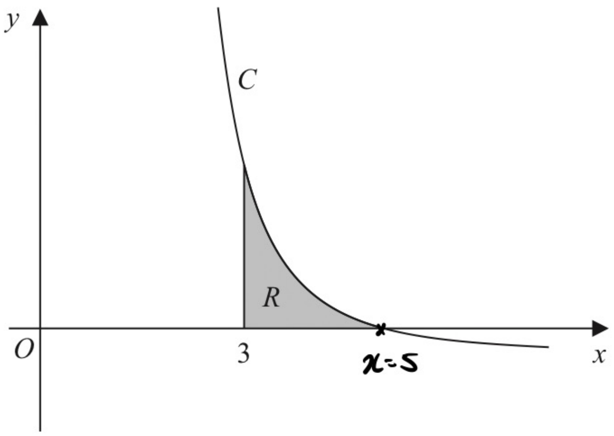

Figure 4

Figure 4 shows a sketch of part of the curve C. The region R, shown shaded in Figure 4, is bounded by the curve C, the x-axis and the line with equation $x = 3$

(b) Show that the exact value of the area of R is $a\ln 2 + b\ln 3$, where a and b are rational constants to be found.

(8)

1

$$\frac{15-3x}{(2x-4)(x+3)} = \frac{1.8}{(2x-4)} - \frac{2.4}{(x+3)}$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int_3^5 \frac{1.8}{(2x-4)} - \frac{2.4}{(x+3)}\,dx = \Big[0.9\ln|2x-4| - 2.4\ln|x+3|\Big]_3^5$$

$f(x) = 2x-4$
$f'(x) = 2$
$\frac{1.8}{2} = 0.9$

$f(x) = x+3$
$f'(x) = 1$
$\frac{2.4}{1} = 2.4$

4

$= 0.9\ln|2(5)-4| - 2.4\ln|5+3| - \big[0.9\ln|2(3)-4| - 2.4\ln|3+3|\big]$

$= 0.9\ln|6| - 2.4\ln|8| - 0.9\ln|2| + 2.4\ln|6|$

using log laws
$\ln(a\times b) = \ln a + \ln b$
$\ln a^k = k\ln a$

$= 3.3\ln|6| - 2.4\ln|8| - 0.9\ln|2|$
$= 3.3(\ln|3| + \ln|2|) - 2.4\ln|2^3| - 0.9\ln|2|$
$= 3.3\ln|3| + 3.3\ln|2| - 7.2\ln|2| - 0.9\ln|2|$
$= 3.3\ln|3| - 4.8\ln|2|$ 1

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7. (a) Use the substitution $u = 1 + \sqrt{x}$ to show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = \int_p^q \frac{2(u-1)^3}{u}\,du$$

where p and q are constants to be found.

(3)

(b) Hence show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = A - B\ln 5$$

where A and B are constants to be found.

(4)

a) $u = 1 + \sqrt{x}$

$u - 1 = \sqrt{x}$

$(u-1)^2 = x$ ①

use the shortened product rule for brackets:

$\frac{d}{dx}(ax+b)^n = a(n-1)(ax+b)^{n-1}$

$\frac{dx}{du} = 2(u-1)$

$\times du$

$\therefore dx = 2(u-1)\,du$

$$\int \frac{x}{1+\sqrt{x}}\,dx = \int \frac{(u-1)^2}{u} \times 2(u-1)\,du = \int \frac{2(u-1)^3}{u}\,du \quad ①$$

$x = 16, \quad u = 1 + \sqrt{16}, \quad u = 5$

$x = 0, \quad u = 1 + \sqrt{0}, \quad u = 1$

These are the new limits

$$\therefore \int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = \int_1^5 \frac{2(u-1)^3}{u}\,du \quad ①$$

where $p = 1$

$q = 5$

expand brackets

b) $\int_0^5 \frac{2(u-1)^3}{u} \, du = 2\int_0^5 \frac{u^3 - 3u^2 + 3u - 1}{u} \, du$

$\int \frac{1}{u} \, du = \ln u$

$= 2\int_0^5 \left(u^2 - 3u + 3 - \frac{1}{u}\right) du$ 1

$= 2\left[\frac{u^3}{3} - \frac{3u^2}{2} + 3u - \ln u\right]_0^5$ 1

$= 2\left[\frac{5^3}{3} - \frac{3(5^2)}{2} + 3(5) - \ln 5 - \left(\frac{1}{3} - \frac{3}{2} + 3 - \ln 1\right)\right]$ 1

$= \frac{104}{3} - 2\ln 5$ 1