Questions

Q1.

Show that

$$\int_0^2 2x\sqrt{x+2}\,dx = \frac{32}{15}\left(2+\sqrt{2}\right)$$

(7)

(Total for question = 7 marks)

Q2.

(a) Use the substitution $x = u^2 + 1$ to show that

$$\int_5^{10} \frac{3\,\mathrm{d}x}{(x-1)\left(3+2\sqrt{x-1}\right)} = \int_p^q \frac{6\,\mathrm{d}u}{u(3+2u)}$$

where p and q are positive constants to be found.

(4)

(b) Hence, using algebraic integration, show that

$$\int_5^{10} \frac{3\,\mathrm{d}x}{(x-1)\left(3+2\sqrt{x-1}\right)} = \ln a$$

where a is a rational constant to be found.

(6)

(Total for question = 10 marks)

Q3.

(a) Use the substitution $u = 1 + \sqrt{x}$ to show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,\mathrm{d}x = \int_p^q \frac{2(u-1)^3}{u}\,\mathrm{d}u$$

where p and q are constants to be found.

(3)

(b) Hence show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,\mathrm{d}x = A - B\ln 5$$

where A and B are constants to be found.

(4)

(Total for question = 7 marks)

Q4.

(a) Use the substitution $u = 4 - \sqrt{h}$ to show that

$$\int \frac{\mathrm{d}h}{4-\sqrt{h}} = -8\ln\left|4-\sqrt{h}\right| - 2\sqrt{h} + k$$

where k is a constant

(6)

A team of scientists is studying a species of slow growing tree.

The rate of change in height of a tree in this species is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.25}\left(4-\sqrt{h}\right)}{20}$$

where h is the height in metres and t is the time, measured in years, after the tree is planted.

(b) Find, according to the model, the range in heights of trees in this species.

(2)

One of these trees is one metre high when it is first planted.

According to the model,

(c) calculate the time this tree would take to reach a height of 12 metres, giving your answer to 3 significant figures.

(7)

(Total for question = 15 marks)

Mark Scheme

Q1.

Question	Scheme for Substitution		Marks	AOs
	Chooses a suitable method for $\int_0^2 2x\sqrt{x+2}\,\mathrm{d}x$ Award for • Using a valid substitution $u = ...$, changing the terms to u's • integrating and using appropriate limts .		M1	3.1a
	Substitution $u = \sqrt{x+2} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = 2u$ oe	Substitution $u = x+2 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = 1$ oe	B1	1.1b
	$\int 2x\sqrt{x+2}\,\mathrm{d}x$ $= \int A(u^2 \pm 2)u^2\,\mathrm{d}u$	$\int 2x\sqrt{x+2}\,\mathrm{d}x$ $= \int A(u \pm 2)\sqrt{u}\,\mathrm{d}u$	M1	1.1b
	$= Pu^5 \pm Qu^3$	$= Su^{\frac{5}{2}} \pm Tu^{\frac{3}{2}}$	dM1	2.1
	$= \frac{4}{5}u^5 - \frac{8}{3}u^3$	$= \frac{4}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}}$	A1	1.1b
	Uses limits 2 and $\sqrt{2}$ the correct way around	Uses limits 4 and 2 the correct way around	ddM1	1.1b
	$= \frac{32}{15}(2+\sqrt{2})$ *		A1*	2.1
			(7)	
				(7 marks)

M1: For attempting to integrate using substitution. Look for
- terms and limits changed to u's. Condone slips and errors/omissions on changing $\mathrm{d}x \to \mathrm{d}u$
- attempted multiplication of terms and raising of at least one power of u by one. Condone slips
- Use of at least the top correct limit. For instance if they go back to x's the limit is 2

B1: For substitution it is for giving the substitution and stating a correct $\frac{\mathrm{d}x}{\mathrm{d}u}$

Eg, $u = \sqrt{x+2} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = 2u$ or equivalent such as $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x+2}}$

M1: It is for attempting to get all aspects of the integral in terms of 'u'.
All terms must be attempted including the $\mathrm{d}x$. You are only condoning slips on signs and coefficients
dM1: It is for using a correct method of expanding and integrating each term (seen at least once) . It is dependent upon the previous M
A1: Correct answer in x or u See scheme
ddM1: Dependent upon the previous M, it is for using the correct limits for their integral, **the correct way around**

A1*: Proceeds correctly to $= \frac{32}{15}(2+\sqrt{2})$. **Note that this is a given answer**

There must be at one least correct intermediate line between $\left[\frac{4}{5}u^5 - \frac{8}{3}u^3\right]_{\sqrt{2}}^{2}$ and $\frac{32}{15}(2+\sqrt{2})$

Question Alt	Scheme for by parts	Marks	AOs
	Chooses a suitable method for $\int_0^2 2x\sqrt{x+2}\,dx$ Award for • using by parts the correct way around • and using limits	M1	3.1a
	$\int\left(\sqrt{x+2}\right)dx = \frac{2}{3}(x+2)^{\frac{3}{2}}$	B1	1.1b
	$\int 2x\sqrt{x+2}\,dx = Ax(x+2)^{\frac{3}{2}} - \int B(x+2)^{\frac{3}{2}}(dx)$	M1	1.1b
	$= Ax(x+2)^{\frac{3}{2}} - C(x+2)^{\frac{5}{2}}$	dM1	2.1
	$= \frac{4}{3}x(x+2)^{\frac{3}{2}} - \frac{8}{15}(x+2)^{\frac{5}{2}}$	A1	1.1b
	Uses limits 2 and 0 the correct way around	ddM1	1.1b
	$= \frac{32}{15}\left(2+\sqrt{2}\right)$	A1*	2.1
		(7)	

M1: For attempting using by parts to solve It is a problem- solving mark and all elements do not have to be correct.

- the formula applied the correct way around. You may condone incorrect attempts at integrating $\sqrt{x+2}$ for this problem solving mark
- further integration, again, this may not be correct, and the use of at least the top limit of 2

B1: For $\int\left(\sqrt{x+2}\right)dx = \frac{2}{3}(x+2)^{\frac{3}{2}}$ oe May be awarded $\int_0^2 2x\sqrt{x+2}\,dx \to x^2 \times \frac{2(x+2)^{\frac{3}{2}}}{3}$

M1: For integration by parts the right way around. Award for $Ax(x+2)^{\frac{3}{2}} - \int B(x+2)^{\frac{3}{2}}(dx)$

dM1: For integrating a second time. Award for $Ax(x+2)^{\frac{3}{2}} - C(x+2)^{\frac{5}{2}}$

A1: $\frac{4}{3}x(x+2)^{\frac{3}{2}} - \frac{8}{15}(x+2)^{\frac{5}{2}}$ which may be un simplified

ddM1: Dependent upon the previous M, it is for using the limits 2 and 0 the **correct way around**

A1*: Proceeds to $= \frac{32}{15}\left(2+\sqrt{2}\right)$. **Note that this is a given answer.**

At least one correct intermediate line must be seen. (See substitution). You would condone missing dx's

Q2.

Question	Scheme	Marks	AOs
(a)	$x = u^2 + 1 \Rightarrow \mathrm{d}x = 2u\mathrm{d}u$ oe	B1	1.1b
	Full substitution $\displaystyle\int \frac{3\mathrm{d}x}{(x-1)(3+2\sqrt{x-1})} = \int \frac{3\times 2u\,\mathrm{d}u}{(u^2+1-1)(3+2u)}$	M1	1.1b
	Finds correct limits e.g. $p = 2, q = 3$	B1	1.1b
	$= \displaystyle\int \frac{3\times 2\cancel{u}\,\mathrm{d}u}{u^{\cancel{2}}(3+2u)} = \int \frac{6\,\mathrm{d}u}{u(3+2u)}$ *	A1*	2.1
		(4)	
(b)	$\dfrac{6}{u(3+2u)} = \dfrac{A}{u} + \dfrac{B}{3+2u} \Rightarrow A = ..., B = ...$	M1	1.1b
	Correct PF. $\dfrac{6}{u(3+2u)} = \dfrac{2}{u} - \dfrac{4}{3+2u}$	A1	1.1b
	$\displaystyle\int \frac{6\,\mathrm{d}u}{u(3+2u)} = 2\ln u - 2\ln(3+2u) \quad (+c)$	dM1 A1ft	3.1a 1.1b
	Uses limits $u = "3", u = "2"$ with some correct ln work leading to $k\ln b$ E.g. $(2\ln 3 - 2\ln 9) - (2\ln 2 - 2\ln 7) = 2\ln\frac{7}{6}$	M1	1.1b
	$\ln\frac{49}{36}$	A1	2.1
		(6)	
		(10 marks)	
Notes: Mark (a) and (b) together as one complete question			

(a)

B1: $\mathrm{d}x = 2u\mathrm{d}u$ or exact equivalent. E.g. $\dfrac{\mathrm{d}x}{\mathrm{d}u} = 2u$, $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{2}(x-1)^{-\frac{1}{2}}$

M1: Attempts a full substitution of $x = u^2 + 1$, including $\mathrm{d}x \to ..u\mathrm{d}u$ to form an integrand in terms of u. Condone slips but there should be an attempt to use the correct substitution on the denominator.

B1: Finds correct limits either states $p = 2, q = 3$ or sight of embedded values as limits to the integral

A1*: Clear reasoning including one fully correct intermediate line, including the integral signs, leading to the given expression ignoring limits. So B1, M1, B0, A1 is possible if the limits are incorrect, omitted or left as 5 and 10.

(b)

M1: Uses correct form of PF leading to values of A and B.

A1: Correct PF $\dfrac{6}{u(3+2u)} = \dfrac{2}{u} - \dfrac{4}{3+2u}$ (Not scored for just the correct values of A and B)

dM1: This is an overall problem solving mark. It is for using the correct PF form and integrating using lns. Look for $P\ln u + Q\ln(3+2u)$

A1ft: Correct integration for their $\dfrac{A}{u} + \dfrac{B}{3+2u} \to A\ln u + \dfrac{B}{2}\ln(3+2u)$ with or without modulus signs

M1: Uses their 2 and 3 as limits, with at least one correct application of the addition law or subtraction law leading to the form $k\ln b$ or $\ln a$. PF's must have been attempted. Condone bracketing slips. Alternatively changing the u's back to x's and use limits of 5 and 10.

A1: Proceeds to $\ln\frac{49}{36}$. Answers without working please send to review.

Q3.

Question	Scheme	Marks	AOs
(a)	$u = 1+\sqrt{x} \Rightarrow x = (u-1)^2 \Rightarrow \frac{dx}{du} = 2(u-1)$ or $u = 1+\sqrt{x} \Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$	B1	1.1b
	$\int \frac{x}{1+\sqrt{x}}dx = \int \frac{(u-1)^2}{u}2(u-1)du$ or $\int \frac{x}{1+\sqrt{x}}dx = \int \frac{x}{u}\times 2x^{\frac{1}{2}}du = \int \frac{2x^{\frac{3}{2}}}{u}du = \int \frac{2(u-1)^3}{u}du$	M1	2.1
	$\int_0^{16} \frac{x}{1+\sqrt{x}}dx = \int_1^5 \frac{2(u-1)^3}{u}du$	A1	1.1b
		(3)	
(b)	$2\int \frac{u^3-3u^2+3u-1}{u}du = 2\int \left(u^2-3u+3-\frac{1}{u}\right)du = ...$	M1	3.1a
	$=(2)\left[\frac{u^3}{3}-\frac{3u^2}{2}+3u-\ln u\right]$	A1	1.1b
	$=2\left[\frac{5^3}{3}-\frac{3(5)^2}{2}+3(5)-\ln 5-\left(\frac{1}{3}-\frac{3}{2}+3-\ln 1\right)\right]$	dM1	2.1
	$=\frac{104}{3}-2\ln 5$	A1	1.1b
		(4)	
			(7 marks)

Notes
(a) B1: Correct expression for $\frac{dx}{du}$ or $\frac{du}{dx}$ $(\text{or } u')$ or dx in terms of du or du in terms of dx M1: Complete method using the given substitution. This needs to be a correct method for their $\frac{dx}{du}$ or $\frac{du}{dx}$ leading to an integral in terms of u only (ignore any limits if present) so for each case you need to see: $\frac{dx}{du} = f(u) \to \int \frac{x}{1+\sqrt{x}}dx = \int \frac{(u-1)^2}{u} f(u)du$ $\frac{du}{dx} = g(x) \to \int \frac{x}{1+\sqrt{x}}dx = \int \frac{x}{u} \times \frac{du}{g(x)} = \int h(u)du$. In this case you can condone slips with coefficients e.g. allow $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \to \int \frac{x}{1+\sqrt{x}}dx = \int \frac{x}{u} \times \frac{x^{\frac{1}{2}}}{2}du = \int h(u)du$ but not $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \to \int \frac{x}{1+\sqrt{x}}dx = \int \frac{x}{u} \times \frac{x^{-\frac{1}{2}}}{2}du = \int h(u)du$ A1: All correct with correct limits and no errors. The "du" must be present but may have been omitted along the way but it must have been seen at least once before the final answer. The limits can be seen as part of the integral or stated separately. (b) M1: Realises the requirement to cube the bracket and divide through by u and makes progress with the integration to obtain at least 3 terms of the required form e.g. 3 from ku^3, ku^2, ku, $k\ln u$ A1: Correct integration. This mark can be scored with the "2" still outside the integral or even if it has been omitted. But if the "2" has been combined with the integrand, the integration must be correct. dM1: Completes the process by applying their "changed" limits and subtracts the right way round **Depends on the first method mark.** A1: Cao (Allow equivalents for $\frac{104}{3}$ e.g. $\frac{208}{6}$)

Q4.

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$\mathrm{d}h = -2(4-u)\,\mathrm{d}u$	B1	This mark is given for finding an expression for dh
	$\int \frac{\mathrm{d}h}{4-\sqrt{h}} = \int \frac{-2(4-u)\,\mathrm{d}u}{4-\sqrt{h}}$	M1	This mark is given for substituting $u = 4-\sqrt{h}$ into the integral
	$= \int -\frac{8}{u} + 2\,\mathrm{d}u$	M1	This mark is given for a method to find a simplified version of the integral
	$-8\ln u + 2u + c$ $= -8\ln(4-\sqrt{h}) + 2(4-\sqrt{h}) + c$	M1	This mark is given for integrating with respect to u to produce an expression in terms of h
		A1	This mark is given for a correct expression for the integral
	$= -8\ln(4-\sqrt{h}) - 2\sqrt{h} + k$	A1	This mark is given for a full proof to arrive at the answer as shown (appreciating that $k = c + 8$)
(b)	$\frac{\mathrm{d}h}{\mathrm{d}t} = 0 \Rightarrow 4-\sqrt{h} = 0$	M1	This mark is given for a setting $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$
	$0 < h < 16$	A1	This mark is given for deducing the range of the heights of the trees for this model

Part	Working or answer an examiner might expect to see	Mark	Notes
(c)	$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{t^{0.25}(4-\sqrt{h})}{20} \Rightarrow \frac{\mathrm{d}h}{(4-\sqrt{h})} = \frac{t^{0.25}\mathrm{d}t}{20}$	B1	This mark is given for separating the variables
	$-8\ln(4-\sqrt{h}) - 2\sqrt{h} + k = \frac{t^{1.25}}{25}$	M1	This mark is given for a method to integrate both sides of the equation
		A1	This mark is given for integrating both sides of the equation correctly
	When $t = 0$ and $h = 1$, $-8\ln 3 - 2 + k = 0$ $k = 2 + 8\ln 3$	M1	This mark is given for substituting values of $t = 0$ and $h = 1$ to find a value for k
	When $h = 12$, $-8\ln(4-\sqrt{12}) - 2\sqrt{12} + 2 + 8\ln 3 = \frac{t^{1.25}}{25}$	M1	This mark is given for a method to find a value for t by substituting $h = 12$ into the equation
	$t^{1.25} = 221.2795 \Rightarrow t = \sqrt[1.25]{221.2795}$	M1	This mark is given for simplifying to find an expression for t
	$t = 75.2$ years	A1	This mark is given for correctly finding the time the tree would take to reach a height of 12 metres
			(Total 15 marks)