Questions

Q1.

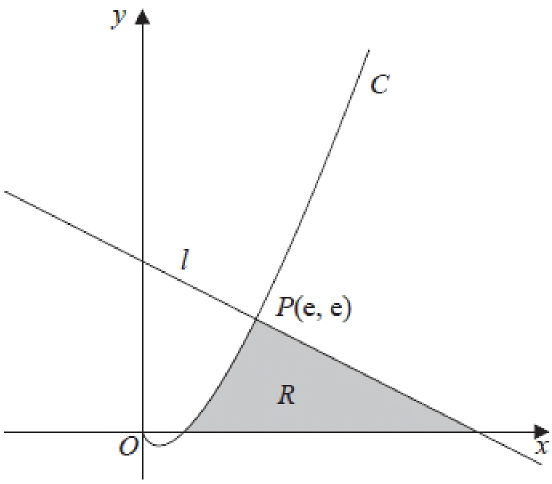

Figure 2

Figure 2 shows a sketch of part of the curve C with equation $y = x \ln x$, $x > 0$

The line l is the normal to C at the point $P(\mathrm{e}, \mathrm{e})$

The region R, shown shaded in Figure 2, is bounded by the curve C, the line l and the x-axis.

Show that the exact area of R is $A\mathrm{e}^2 + B$ where A and B are rational numbers to be found.

(10)

(Total for question = 10 marks)

Q2.

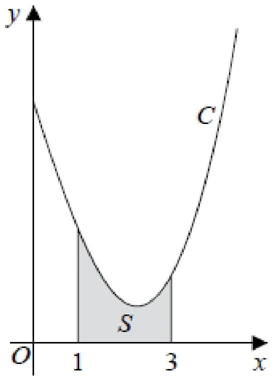

Figure 4

Figure 4 shows a sketch of part of the curve *C* with equation

$$y = \frac{x^2 \ln x}{3} - 2x + 5, \quad x > 0$$

The finite region *S*, shown shaded in Figure 4, is bounded by the curve *C*, the line with equation *x* = 1, the *x*-axis and the line with equation *x* = 3

The table below shows corresponding values of *x* and *y* with the values of *y* given to 4 decimal places as appropriate.

x	1	1.5	2	2.5	3
y	3	2.3041	1.9242	1.9089	2.2958

(a) Use the trapezium rule, with all the values of *y* in the table, to obtain an estimate for the area of *S*, giving your answer to 3 decimal places.

(3)

(b) Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of *S*.

(1)

(c) Show that the exact area of *S* can be written in the form $\frac{a}{b}$ + ln*c*, where *a*, *b* and *c* are integers to be found.

(6)

(*In part c*, solutions based entirely on graphical or numerical methods are not acceptable.)

(Total for question = 10 marks)

Mark Scheme

Q1.

Question	Scheme	Marks	AOs
	$C: y = x\ln x$; l is a normal to C at $P(\text{e}, \text{e})$ Let x_A be the x-coordinate of where l cuts the x-axis		
	$\frac{\text{d}y}{\text{d}x} = \ln x + x\left(\frac{1}{x}\right)$ $\{= 1 + \ln x\}$	M1 A1	2.1 1.1b
	$x = \text{e},\ m_T = 2 \Rightarrow m_N = -\frac{1}{2} \Rightarrow y - \text{e} = -\frac{1}{2}(x - \text{e})$ $y = 0 \Rightarrow -\text{e} = -\frac{1}{2}(x - \text{e}) \Rightarrow x = \ldots$	M1	3.1a
	l meets x-axis at $x = 3\text{e}$ (allow $x = 2\text{e} + \text{e}\ln\text{e}$)	A1	1.1b
	{Areas:} **either** $\int_1^{\text{e}} x\ln x\,\text{d}x = [\ \ldots\ ]_1^{\text{e}} = \ldots$ **or** $\frac{1}{2}((\text{their } x_A) - \text{e})\text{e}$	M1	2.1
	$\left\{\int x\ln x\,\text{d}x = \right\}\ \frac{1}{2}x^2\ln x - \int \frac{1}{x}\cdot\left(\frac{x^2}{2}\right)\{\text{d}x\}$	M1	2.1
	$\left\{= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,\{\text{d}x\}\right\} = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$	dM1 A1	1.1b 1.1b
	$\text{Area}(R_1) = \int_1^{\text{e}} x\ln x\,\text{d}x = [\ \ldots\ ]_1^{\text{e}} = \ldots$; $\text{Area}(R_2) = \frac{1}{2}((\text{their } x_A) - \text{e})\text{e}$ and so, $\text{Area}(R) = \text{Area}(R_1) + \text{Area}(R_2)$ $\{= \frac{1}{4}\text{e}^2 + \frac{1}{4} + \text{e}^2\}$	M1	3.1a
	$\text{Area}(R) = \frac{5}{4}\text{e}^2 + \frac{1}{4}$	A1	1.1b
		(10)	

Notes for Question	
M1:	Differentiates by using the product rule to give $\ln x + x(\text{their } g'(x))$, where $g(x) = \ln x$
A1:	Correct differentiation of $y = x\ln x$, which can be un-simplified or simplified
M1:	Complete strategy to find the x coordinate where their normal to C at $P(e, e)$ meets the x-axis i.e. Sets $y = 0$ in $y - e = m_N(x - e)$ to find $x = ...$
Note:	m_T is found by using calculus and $m_N \neq m_T$
A1:	l meets x-axis at $x = 3e$, allowing un-simplified values for x such as $x = 2e + e\ln e$
Note:	Allow $x =$ awrt 8.15
M1:	Scored for either • Area under curve $= \int_1^e x\ln x\,dx = [\ ...\ ]_1^e = ...$, with limits of e and 1 and some attempt to substitute these and subtract • or Area under line $= \frac{1}{2}((\text{their } x_A) - e)e$, with a valid attempt to find x_A
M1:	Integration by parts the correct way around to give $Ax^2\ln x - \int B\left(\frac{x^2}{x}\right)\{dx\}$; $A \neq 0, B > 0$
dM1:	**dependent on the previous M mark** Integrates the second term to give $\pm\lambda x^2$; $\lambda \neq 0$
A1:	$\frac{1}{2}x^2\ln x - \frac{1}{4}x^2$
M1:	Complete strategy of finding the area of R by finding the sum of two key areas. See scheme.
A1:	$\frac{5}{4}e^2 + \frac{1}{4}$
Note:	$\text{Area}(R_2)$ can also be found by integrating the line l between limits of e and their x_A i.e. $\text{Area}(R_2) = \int_e^{\text{their } x_A} \left(-\frac{1}{2}x + \frac{3}{2}e\right) dx = [\ ...\ ]_e^{\text{their } x_A} = ...$
Note:	<u>**Calculator approach with no algebra, differentiation or integration seen:**</u> • Finding l cuts through the x-axis at awrt 8.15 is 2nd M1 2nd A1 • Finding area between curve and the x-axis between $x = 1$ and $x = e$ to give awrt 2.10 is 3rd M1 • Using the above information (must be seen) to apply $\text{Area}(R) = 2.0972... + 7.3890... = 9.4862...$ is final M1 Therefore, a maximum of 4 marks out of the 10 available.

Q2.

Question	Scheme	Marks	AOs
(a)	Uses or implies $h = 0.5$	B1	1.1b
	For correct form of the trapezium rule =	M1	1.1b
	$\frac{0.5}{2}\{3+2.2958+2(2.3041+1.9242+1.9089)\}=4.393$	A1	1.1b
		(3)	
(b)	Any valid statement reason, for example • Increase the number of strips • Decrease the width of the strips • Use more trapezia	B1	2.4
		(1)	
(c)	For integration by parts on $\int x^2 \ln x \, dx$	M1	2.1
	$=\frac{x^3}{3}\ln x - \int \frac{x^2}{3} dx$	A1	1.1b
	$\int -2x+5 \, dx = -x^2+5x \quad (+c)$	B1	1.1b
	All integration attempted and limits used Area of $S = \int_1^3 \frac{x^2 \ln x}{3} - 2x + 5 \, dx = \left[\frac{x^3}{9}\ln x - \frac{x^3}{27} - x^2 + 5x\right]_{x=1}^{x=3}$	M1	2.1
	Uses correct ln laws, simplifies and writes in required form	M1	2.1
	Area of $S = \frac{28}{27} + \ln 27 \quad (a = 28, b = 27, c = 27)$	A1	1.1b
		(6)	
			(10 marks)

Notes:	
(a)	
B1:	States or uses the strip width $h = 0.5$. This can be implied by the sight of $\frac{0.5}{2}\{...\}$ in the trapezium rule
M1:	For the correct form of the bracket in the trapezium rule. Must be y values rather than x values $\{\text{first } y \text{ value} + \text{last } y \text{ value} + 2\times(\text{sum of other } y \text{ values})\}$
A1:	4.393
(b)	
B1:	See scheme
(c)	
M1:	Uses integration by parts the right way around. Look for $\int x^2 \ln x \, dx = Ax^3 \ln x - \int Bx^2 \, dx$
A1:	$\frac{x^3}{3}\ln x - \int \frac{x^2}{3} \, dx$
B1:	Integrates the $-2x+5$ term correctly $= -x^2 + 5x$
M1:	All integration completed and limits used
M1:	Simplifies using ln law(s) to a form $\frac{a}{b} + \ln c$
A1:	Correct answer only $\frac{28}{27} + \ln 27$