## Questions

**Q1.**

The curve *C* with equation

$$y = \frac{p - 3x}{(2x - q)(x + 3)} \qquad x \in \mathbb{R}, x \neq -3, x \neq 2$$

where *p* and *q* are constants, passes through the point $\left(3, \frac{1}{2}\right)$ and has two vertical asymptotes with equations $x = 2$ and $x = -3$

(a) (i) Explain why you can deduce that $q = 4$

(ii) Show that $p = 15$

**(3)**

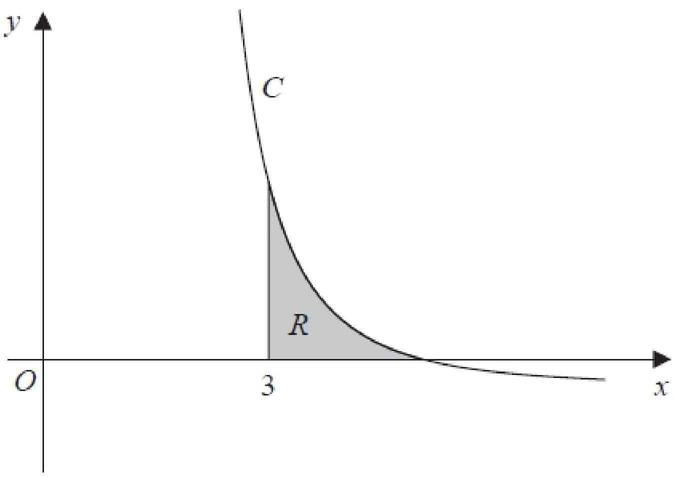


**Figure 4**

Figure 4 shows a sketch of part of the curve *C*. The region *R*, shown shaded in Figure 4, is bounded by the curve *C*, the *x*-axis and the line with equation $x = 3$

(b) Show that the exact value of the area of *R* is $a\ln 2 + b\ln 3$, where *a* and *b* are rational constants to be found.

**(8)**

**(Total for question = 11 marks)**

**Q2.**

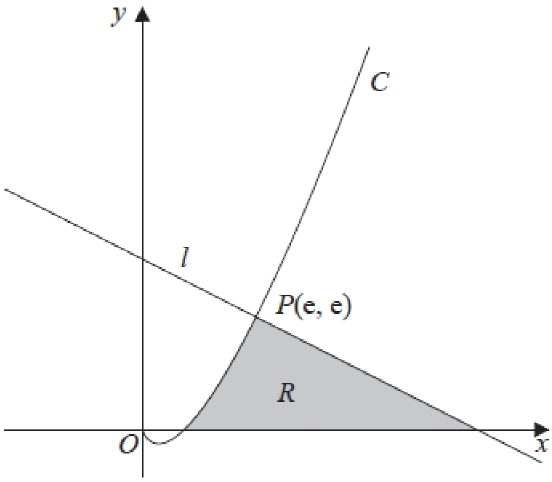


**Figure 2**

Figure 2 shows a sketch of part of the curve $C$ with equation $y = x \ln x, \quad x > 0$

The line $l$ is the normal to $C$ at the point $P(\mathrm{e}, \mathrm{e})$

The region $R$, shown shaded in Figure 2, is bounded by the curve $C$, the line $l$ and the $x$-axis.

Show that the exact area of $R$ is $A\mathrm{e}^2 + B$ where $A$ and $B$ are rational numbers to be found.

**(10)**

**(Total for question = 10 marks)**

**Q3.**

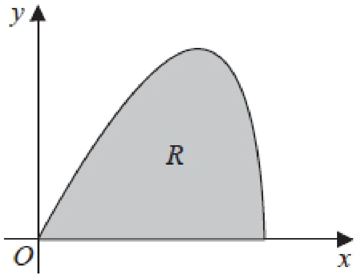


**Figure 3**

The curve shown in Figure 3 has parametric equations

$$x = 6\sin t \qquad y = 5\sin 2t \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

The region *R*, shown shaded in Figure 3, is bounded by the curve and the *x*-axis.

(a) (i) Show that the area of *R* is given by $\int_0^{\frac{\pi}{2}} 60\sin t\cos^2 t\,dt$

**(3)**

(ii) Hence show, by algebraic integration, that the area of *R* is exactly 20

**(3)**

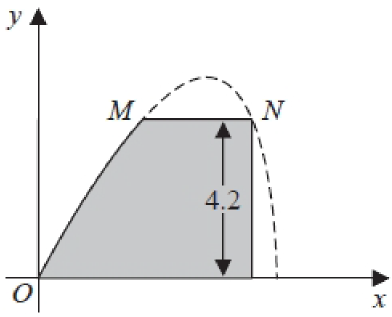


**Figure 4**

Part of the curve is used to model the profile of a small dam, shown shaded in Figure 4.
Using the model and given that

- *x* and *y* are in metres
- the vertical wall of the dam is 4.2 metres high
- there is a horizontal walkway of width *MN* along the top of the dam

(b) calculate the width of the walkway.

**(5)**

**(Total for question = 11 marks)**

**Q4.**

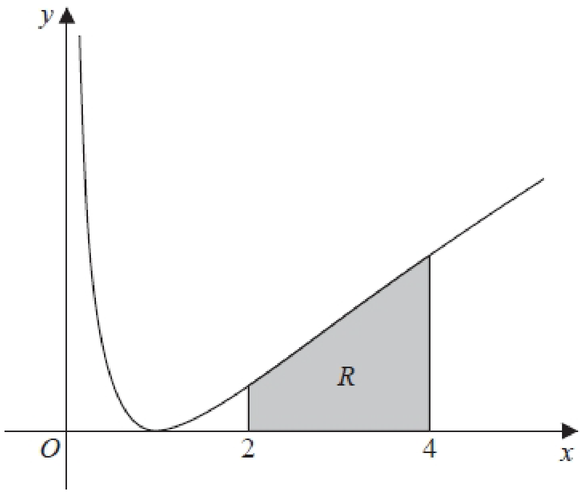


**Figure 2**

Figure 2 shows a sketch of part of the curve with equation

$$y = (\ln x)^2 \qquad x > 0$$

The finite region *R*, shown shaded in Figure 2, is bounded by the curve, the line with equation *x* = 2, the *x*-axis and the line with equation *x* = 4

The table below shows corresponding values of *x* and *y*, with the values of *y* given to 4 decimal places.

| $x$ | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|
| $y$ | 0.4805 | 0.8396 | 1.2069 | 1.5694 | 1.9218 |

(a) Use the trapezium rule, with all the values of *y* in the table, to obtain an estimate for the area of *R*, giving your answer to 3 significant figures.

**(3)**

(b) Use algebraic integration to find the exact area of *R*, giving your answer in the form

$$y = a(\ln 2)^2 + b\ln 2 + c$$

where *a*, *b* and *c* are integers to be found.

**(5)**

**(Total for question = 8 marks)**

## Mark Scheme

**Q1.**

| Part | Working or answer an examiner might expect to see | Mark | Notes |
|---|---|---|---|
| (a) | The asymptote is found where $2x - q = 0$<br>Hence $q = 4$ | B1 | This mark is given for explaining that the asymptote at $x = 2$ is a solution of $2x - q = 0$ |
| | $y = \dfrac{p-3x}{(2x-4)(x+3)}$<br>$\dfrac{1}{2} = \dfrac{p-9}{(6-4)(3+3)}$ | M1 | This mark is given for substituting $x = 3$, $y = \dfrac{1}{2}$ (and $q = 4$) |
| (b) | $\dfrac{15-3x}{(2x-4)(x+3)} = \dfrac{A}{(2x-4)} + \dfrac{B}{(x+3)}$ | M1 | This mark is given for a method to use partial fractions |
| | $= \dfrac{1.8}{(2x-4)} - \dfrac{2.4}{(x+3)}$ | M1 | This mark is given for finding values for $A$ and $B$ |
| | $= \dfrac{0.9}{(x-2)} - \dfrac{2.4}{(x+3)}$ | A1 | This mark is given for a fully simplified expression |
| | $I = \displaystyle\int \dfrac{15-3x}{(2x-4)(x+3)}\,dx$<br>$= m \ln(2x-4) + n \ln(x+3)$ | M1 | This mark is given for a method to integrate to find the area of $R$ |
| | $= 0.9 \ln(2x-4) + 2.4 \ln(x+3)$ | A1 | This mark is given for a correct expression for the area of $R$ |
| | Area $R = \left[0.9\ln(2x-4) - 2.4\ln(x+3)\right]_3^5$ | M1 | This mark is given for deducing an expression for the area of $R$<br>($y = 0$ when $x = 5$) |
| | $= [0.9 \ln 6 - 2.4 \ln 8] - [0.9 \ln 2 - 2.4 \ln 6]$<br>$= [0.9 \ln 6 + 2.4 \ln 6] - [7.2 \ln 2 + 0.9 \ln 2]$<br>$= 3.3 \ln 6 - 8.1 \ln 2$<br>$= 3.3 \ln 3 + 3.3 \ln 2 - 8.1 \ln 2$ | M1 | This mark is given for a method to find the exact area of $R$ |
| | $= 3.3 \ln 3 - 4.8 \ln 2$ | A1 | This mark is given for a correct value of the area of $R$ with $a = 3.3$ and $b = 4.8$ |

**Q2.**

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| | $C: y = x\ln x$; $l$ is a normal to $C$ at $P(\text{e}, \text{e})$<br>Let $x_A$ be the $x$-coordinate of where $l$ cuts the $x$-axis | | |
| | $\frac{\text{d}y}{\text{d}x} = \ln x + x\left(\frac{1}{x}\right) \quad \{= 1 + \ln x\}$ | M1<br>A1 | 2.1<br>1.1b |
| | $x = \text{e},\ m_T = 2 \Rightarrow m_N = -\frac{1}{2} \Rightarrow y - \text{e} = -\frac{1}{2}(x - \text{e})$<br>$y = 0 \Rightarrow -\text{e} = -\frac{1}{2}(x - \text{e}) \Rightarrow x = \ldots$ | M1 | 3.1a |
| | $l$ meets $x$-axis at $x = 3\text{e}$ (allow $x = 2\text{e} + \text{e}\ln\text{e}$) | A1 | 1.1b |
| | {Areas:} **either** $\int_1^{\text{e}} x\ln x\,\text{d}x = [\ \ldots\ ]_1^{\text{e}} = \ldots$ **or** $\frac{1}{2}((\text{their } x_A) - \text{e})\text{e}$ | M1 | 2.1 |
| | $\left\{\int x\ln x\,\text{d}x = \right\}\ \frac{1}{2}x^2\ln x - \int \frac{1}{x}.\left(\frac{x^2}{2}\right)\{\text{d}x\}$ | M1 | 2.1 |
| | $\left\{= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,\{\text{d}x\}\right\} = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$ | dM1<br>A1 | 1.1b<br>1.1b |
| | $\text{Area}(R_1) = \int_1^{\text{e}} x\ln x\,\text{d}x = [\ \ldots\ ]_1^{\text{e}} = \ldots$ ; $\text{Area}(R_2) = \frac{1}{2}((\text{their } x_A) - \text{e})\text{e}$<br>and so, $\text{Area}(R) = \text{Area}(R_1) + \text{Area}(R_2) \quad \{= \frac{1}{4}\text{e}^2 + \frac{1}{4} + \text{e}^2\}$ | M1 | 3.1a |
| | $\text{Area}(R) = \frac{5}{4}\text{e}^2 + \frac{1}{4}$ | A1 | 1.1b |
| | | (10) | |

| Notes for Question | |
|---|---|
| **M1:** | Differentiates by using the product rule to give $\ln x + x(\text{their } g'(x))$, where $g(x) = \ln x$ |
| **A1:** | Correct differentiation of $y = x\ln x$, which can be un-simplified or simplified |
| **M1:** | Complete strategy to find the $x$ coordinate where their normal to $C$ at $P(e, e)$ meets the $x$-axis<br>i.e. Sets $y = 0$ in $y - e = m_N(x - e)$ to find $x = ...$ |
| **Note:** | $m_T$ is found by using calculus and $m_N \neq m_T$ |
| **A1:** | $l$ meets $x$-axis at $x = 3e$, allowing un-simplified values for $x$ such as $x = 2e + e\ln e$ |
| **Note:** | Allow $x =$ awrt 8.15 |
| **M1:** | Scored for either<br>• Area under curve $= \int_1^e x\ln x \, dx = [\ ...\ ]_1^e = ...$, with limits of e and 1 and some attempt to substitute these and subtract<br>• or Area under line $= \frac{1}{2}((\text{their } x_A) - e)e$, with a valid attempt to find $x_A$ |
| **M1:** | Integration by parts the correct way around to give $Ax^2\ln x - \int B\left(\frac{x^2}{x}\right)\{dx\}$; $A \neq 0, B > 0$ |
| **dM1:** | **dependent on the previous M mark**<br>Integrates the second term to give $\pm\lambda x^2$; $\lambda \neq 0$ |
| **A1:** | $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2$ |
| **M1:** | Complete strategy of finding the area of $R$ by finding the sum of two key areas. See scheme. |
| **A1:** | $\frac{5}{4}e^2 + \frac{1}{4}$ |
| **Note:** | $\text{Area}(R_2)$ can also be found by integrating the line $l$ between limits of e and their $x_A$<br>i.e. $\text{Area}(R_2) = \int_e^{\text{their } x_A} \left(-\frac{1}{2}x + \frac{3}{2}e\right) dx = [\ ...\ ]_e^{\text{their } x_A} = ...$ |
| **Note:** | <u>**Calculator approach with no algebra, differentiation or integration seen:**</u><br>• Finding $l$ cuts through the $x$-axis at awrt 8.15 is 2nd M1 2nd A1<br>• Finding area between curve and the $x$-axis between $x = 1$ and $x = e$ to give awrt 2.10 is 3rd M1<br>• Using the above information (must be seen) to apply $\text{Area}(R) = 2.0972... + 7.3890... = 9.4862...$ is final M1<br>Therefore, a maximum of 4 marks out of the 10 available. |

## Q3.

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a)(i) | $y\times\frac{dx}{dt}=5\sin 2t\times 6\cos t$ or $5\times 2\sin t\cos t\times 6\cos t$ | M1 | 1.2 |
| | | | |
| | (Area =) $\int 5\sin 2t\times 6\cos t\,dt=\int 5\times 2\sin t\cos t\times 6\cos t\,dt$ or $\int 5\sin 2t\times 6\cos t\,dt=\int 60\sin t\cos^2 t\,dt$ | dM1 | 1.1b |
| | (Area =) $\int_0^{\frac{\pi}{2}} 60\sin t\cos^2 t\,dt$ * | A1* | 2.1* |
| | | (3) | |
| (a)(ii) | $\int 60\sin t\cos^2 t\,dt=-20\cos^3 t$ | M1<br>A1 | 1.1b<br>1.1b |
| | Area $=\left[-20\cos^3 t\right]_0^{\frac{\pi}{2}}=0-(-20)=20$ * | A1* | 2.1 |
| | | (3) | |
| (b) | $5\sin 2t=4.2\Rightarrow \sin 2t=\frac{4.2}{5}$ | M1 | 3.4 |
| | $t=0.4986...,1.072...$ | A1 | 1.1b |
| | Attempts to finds the $x$ values at both $t$ values | dM1 | 3.4 |
| | $t=0.4986...\Rightarrow x=2.869...$<br>$t=1.072\Rightarrow x=5.269...$ | A1 | 1.1b |
| | Width of path = 2.40 metres | A1 | 3.2a |
| | | (5) | |
| | | | (11 marks) |

**Notes:**

**(a)(i)**

**M1:** Attempts to multiply $y$ by $\frac{dx}{dt}$ to obtain $A\sin 2t\cos t$ but may apply $\sin 2t=2\sin t\cos t$ here

**dM1:** Attempts to use $\sin 2t=2\sin t\cos t$ within an integral which may be implied by

e.g. $A\int \sin 2t\times\cos t\,dt=\int k\sin t\cos^2 t\,dt$

**A1*:** Fully correct work leading to the given answer.
**This must include** $\sin 2t=2\sin t\cos t$ or e.g. $5\sin 2t=10\sin t\cos t$ seen explicitly in their proof and a correct intermediate line that includes an integral sign and the "d$t$"
**Allow the limits to just "appear" in the final answer e.g. working need not be shown for the limits.**

**(a)(ii)**

**M1:** Obtains $\int 60\sin t\cos^2 t\,dt=k\cos^3 t$. This may be attempted via a substitution of $u=\cos t$ to obtain

$\int 60\sin t\cos^2 t\,dt=ku^3$

**A1:** Correct integration $-20\cos^3 t$ or equivalent e.g. $-20u^3$
**A1*:** Rigorous proof with all aspects correct including the correct limits and the $0-(-20)$ and

not just: $-20\cos^3\frac{\pi}{2}-\left(-20\cos^3 0\right)=20$

**(b)**

**M1:** Uses the given model and attempts to find value(s) of $t$ when $\sin 2t=\frac{4.2}{5}$. Look for $2t=\sin^{-1}\frac{4.2}{5}\Rightarrow t=...$

**A1:** At least one correct value for $t$, correct to 2 dp. FYI $t=0.4986...,1.072...$ or in degrees $t=28.57...,61.42...$

**dM1:** Attempts to find **TWO** distinct values of $x$ when $\sin 2t = \frac{4.2}{5}$. Condone poor trig work and allow this mark if 2 values of $x$ are attempted from 2 values of $t$.
**A1:** Both values correct to 2 dp. NB $x = 2.869\ldots,\ 5.269\ldots$

**Or may take Cartesian approach**

$$5\sin 2t = 4.2 \Rightarrow 10\sin t\cos t = 4.2 \Rightarrow 10\frac{x}{6}\sqrt{1-\frac{x^2}{36}} = 4.2 \Rightarrow x^4 - 36x^2 + 228.6144 = 0 \Rightarrow x = 2.869...,5.269...$$

M1: For converting to Cartesian form A1: Correct quartic M1: Solves quartic A1: Correct values

**A1:** 2.40 metres or 240 cm
Allow awrt 2.40 m or allow 2.4m (not awrt 2.4 m) and allow awrt 240 cm. **Units are required.**

## Q4.

| Question | Scheme | Marks | AOs |
|---|---|---|---|
| (a) | $h = 0.5$ | B1 | 1.1b |
| | $A \approx \frac{1}{2}\times\frac{1}{2}\{0.4805 + 1.9218 + 2(0.8396 + 1.2069 + 1.5694)\}$ | M1 | 1.1b |
| | $= 2.41$ | A1 | 1.1b |
| | | (3) | |
| (b) | $\int(\ln x)^2\ \mathrm{d}x = x(\ln x)^2 - \int x \times \frac{2\ln x}{x}\ \mathrm{d}x$ | M1<br>A1 | 3.1a<br>1.1b |
| | $= x(\ln x)^2 - 2\int \ln x\ \mathrm{d}x = x(\ln x)^2 - 2\left(x\ln x - \int \mathrm{d}x\right)$<br>$= x(\ln x)^2 - 2\int \ln x\ \mathrm{d}x = x(\ln x)^2 - 2x\ln x + 2x$ | dM1 | 2.1 |
| | $\int_2^4 (\ln x)^2\ \mathrm{d}x = \left[x(\ln x)^2 - 2x\ln x + 2x\right]_2^4$<br>$= 4(\ln 4)^2 - 2\times 4\ln 4 + 2\times 4 - \left(2(\ln 2)^2 - 2\times 2\ln 2 + 2\times 2\right)$<br>$= 4(2\ln 2)^2 - 16\ln 2 + 8 - 2(\ln 2)^2 + 4\ln 2 - 4$ | ddM1 | 2.1 |
| | $= 14(\ln 2)^2 - 12\ln 2 + 4$ | A1 | 1.1b |
| | | (5) | |
| | | | (8 marks) |
| | **Notes** | | |

(a)

B1: Correct strip width. May be implied by $\frac{1}{2}\times\frac{1}{2}\{....\}$ or $\frac{1}{4}\times\{....\}$

M1: Correct application of the trapezium rule.

Look for $\frac{1}{2}\times"h"\{0.4805+1.9218+2(0.8396+1.2069+1.5694)\}$ condoning slips in the digits.

The bracketing must be correct but it is implied by awrt 2.41

A1: 2.41 only. This is not awrt

(b)

M1: Attempts parts the correct way round to achieve $\alpha x(\ln x)^2-\beta\int\ln x\,\mathrm{d}x$ o.e.

May be unsimplified (see scheme).

Watch for candidates who know or learn $\int\ln x\,\mathrm{d}x = x\ln x - x$

who may write $\int(\ln x)^2\,\mathrm{d}x=\int(\ln x)(\ln x)\,\mathrm{d}x=\ln x(x\ln x-x)-\int\frac{x\ln x-x}{x}\,\mathrm{d}x$

A1: Correct expression which may be unsimplified

dM1: Attempts parts again to (only condone coefficient errors) to achieve $\alpha x(\ln x)^2-\beta x\ln x\pm\gamma x$ o.e.

ddM1: Applies the limits 4 and 2 to an expression of the form $\pm\alpha x(\ln x)^2\pm\beta x\ln x\pm\gamma x$, subtracts and applies $\ln 4 = 2\ln 2$ at least once. Both M's must have been awarded

A1: Correct answer

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It is possible to do $\int(\ln x)^2\,\mathrm{d}x$ via a substitution $u=\ln x$ but it is very similar.

M1 A1, dM1: $\int u^2\mathrm{e}^u\,\mathrm{d}u = u^2\mathrm{e}^u-\int 2u\mathrm{e}^u\,\mathrm{d}u, = u^2\mathrm{e}^u-2u\mathrm{e}^u\pm 2\mathrm{e}^u$

ddM1: Applies appropriate limits and uses $\ln 4 = 2\ln 2$ at least once to an expression of the form $u^2\mathrm{e}^u-\beta u\mathrm{e}^u\pm\gamma\mathrm{e}^u$ Both M's must have been awarded