Integration Cheat Sheet

Edexcel Pure Year 1

Integration is the reverse of differentiation. We can use integration to find areas bounded between a curve and the coordinate axes.

Notation

The $\int$ symbol is used to represent integration. Since integration is the reverse of differentiation, we have that

- If $\frac{dy}{dx} = f(x)$ then $y = \int f(x)\, dx$

The dx means that we are integrating with respect to x

The integral sign

The expression we want to integrate goes between the integral sign and the dx. This is known as the integrand.

Indefinite integrals

In this chapter, you will only learn how to integrate functions of the form x^n, where n is a constant and $n \neq -1$. To integrate functions of this form, you can use the following result:

- $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$

The " $+c$" is known as the constant of integration. To see why we need to add this constant to our result, consider the following three functions:

- $y = x^2 + 2$
- $y = x^2$
- $y = x^2 - 9$

If we differentiate the above functions, the result is the same: $\frac{dy}{dx} = 2x$ because the constant term disappears upon differentiating. But since integration is the reverse of differentiation, we should be able to integrate $2x$ and get back to whichever of those functions we started with. To allow for this, we must add the unknown constant of integration, c, to our end result.

The above process is known as indefinite integration.

Definite integrals

A definite integral is one where the integral is bounded between two limits. The main difference between a definite integral and an indefinite integral is that a definite integral will yield a numerical value, while an indefinite integral will yield a function. To calculate a definite integral:

- $\int_a^b f'(x)\, dx = [f(x)]_a^b = f(b) - f(a)$

Example 1: Evaluate $\int_2^3 x^2\, dx$.

We write the result in square brackets, with the limits outside

$$\int_2^3 x^2\, dx = \left[\frac{x^3}{3}\right]_2^3 = \left[\frac{3^3}{3}\right] - \left[\frac{2^3}{3}\right] = 9 - \frac{8}{3} = \frac{19}{3}$$

The limits of the integral are 2 and 3, written on the top and bottom of the integral sign.

We substitute our limits into the result, and simplify,

Integrating polynomials

You can use the following two properties of integrals to integrate expressions where there are multiple terms of the form ax^n

- $\int kx^n dx = k\int x^n dx = k\left(\frac{x^{n+1}}{n+1}\right) + c$ for any constant k.
- $\int x^n + x^m\, dx = \int x^n\, dx + \int x^m\, dx$

Here is an example showing how we use these properties in practice:

Example 2: Evaluate $\int 2x^3 - \frac{4}{x^2} + 4x\, dx$

We can split up the integral as follows:

$$\Rightarrow 2\int x^3\, dx + (-4)\int x^{-2}\, dx + 4\int x\, dx$$

$$\Rightarrow 2\left[\frac{x^4}{4}\right] + c - 4\left[\frac{x^{-1}}{-1}\right] + d + 4\left[\frac{x^2}{2}\right] + e$$

$$\Rightarrow \frac{x^4}{2} + \frac{4}{x} + 2x^2 + k$$

Each of the integrals yields an integration constant (c, d and e), so we can denote their total as one constant, k.

Finding the constant of integration

You can find the constant of integration if you are given a point the curve passes through or the value of the function at a given point.

Example 3: The curve C with equation $y = f(x)$ passes through the point (4,5). Given that $f'(x) = \frac{x^2 - 4x}{\sqrt{x}}$, find the equation of C.

$$f'(x) = \frac{x^2}{x^{0.5}} - 4x^{0.5} = x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$$

$$\therefore f(x) = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{4x^{\frac{3}{2}}}{\frac{3}{2}} + c \Rightarrow y = \frac{2}{5}x^{\frac{5}{2}} - \frac{8}{3}x^{\frac{3}{2}} + c$$

But we are told that at $x = 4, y = 5$. Substituting these values:

$$5 = \frac{2}{5}(4)^{\frac{5}{2}} - \frac{8}{3}(4)^{\frac{3}{2}} + c$$

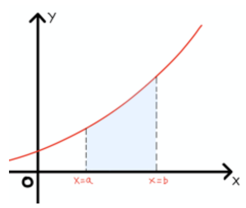
$$5 = \frac{64}{5} - \frac{64}{3} + c \implies c = \frac{203}{15} \qquad \therefore\ y = \frac{2}{5}x^{\frac{5}{2}} - \frac{8}{3}x^{\frac{3}{2}} + \frac{203}{15}$$

Finding Areas

You need to be able to use definite integration to find the area bounded between a curve and the x-axis.

- The area between a curve $y = f(x)$, the lines $x = a$, $x = b$ and the x-axis is given by

$$Area = \int_a^b f(x)\ dx$$



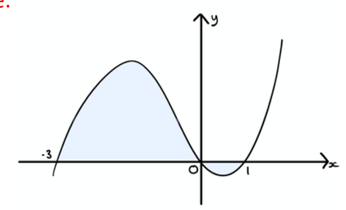
Areas under the x-axis

When integrating over an interval where the curve is below the x-axis, the resultant area will be negative. As a result, you need to take extra care when finding areas under curves which are not strictly positive.

- When integrating over an interval where the curve is both above and below the x-axis, you should split the integral up into separate regions where the function is strictly positive or negative in each.

Example 4: Find the area bounded between the curve with equation $y = x(x-1)(x+3)$ and the x-axis over the interval $-3 < x < 1$.

We start by sketching the curve:



We see that the curve is strictly positive for $-3 < x < 0$ and strictly negative for $0 < x < 1$. We must find the area of these regions separately and add the results together:

Area required = $\int_{-3}^{0} y\, dx + \int_0^1 y\, dx$

$$\int y\, dx = \int x^3 + 2x^2 - 3x\, dx = \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]$$

so $\int_{-3}^{0} y\, dx = (0) - \left(\frac{81}{4} - \frac{2(27)}{3} - \frac{3(9)}{2}\right) = \frac{45}{4}$

and $\int_0^1 y\, dx = \left(\frac{1}{4} + \frac{2}{3} - \frac{3}{2}\right) - (0) = -\frac{7}{12}$

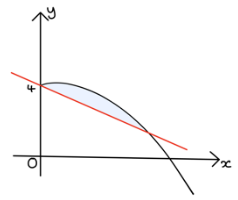
so the required area is $\frac{45}{4} + \frac{7}{12} = \frac{71}{6}$

If we tried to calculate the area using just one definite integral, the negative and positive area would partly cancel out giving us an incorrect answer.

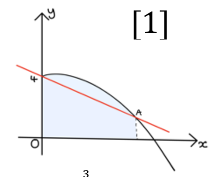
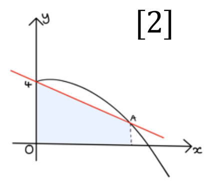
More complicated areas

You may need to combine triangles, trapeziums and direct integrals to calculate more complicated areas. A common type of problem is one where you must find the area bounded between a curve and a line.

Example 5: The diagram shows part of the curve with equation $y = 3\sqrt{x} - \sqrt{x^3} + 4$ and the line with equation $y = 4 - \frac{1}{2}x$. Given that the line and the curve cross at the point $A(4, 2)$, find the area of the finite region bounded by the curve and the line.



To find the required area, we can subtract the area of the trapezium made by the line and the axes from the area under the curve:

Required area = [1]  − [2] 

Integrating the curve. Note that $\sqrt{x^3} = x^{\frac{3}{2}}$

$$[1] = \int_0^4 3\sqrt{x} - \sqrt{x^3} + 4\, dx = \left[\frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 4x\right]_0^4 = \left(2\left(4^{\frac{3}{2}}\right) - \frac{2}{5}(4)^{\frac{5}{2}} + 4(4)\right) - (0) = \frac{96}{5}$$

Substituting the limits

$$[2] = Area\ of\ trapezium = \frac{(a+b)h}{2} = \frac{(4+2)(4)}{2} = 12$$

$$\therefore Area\ required = \frac{96}{5} - 12 = \frac{36}{5}$$