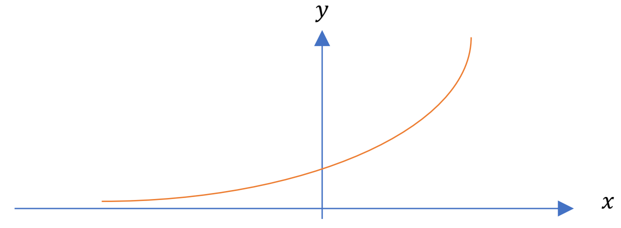
Exponentials and Logarithms Cheat Sheet

Exponential functions

Functions of the form $f(x) = a^x$, where a is a constant, are called exponential functions. You should become familiar with these functions and the shapes of their graphs.
For instance, table below shows an example of values for $y = 2^x$.

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

The graph of $y = 2^x$ is a smooth curve that looks like this:



$y = e^x$

Exponential functions of the form $f(x) = a^x$ have a special property. The graphs of their gradient functions are a similar shape to the graphs of the function themselves. When the value of a is approximately equal to 2.71878, the gradient function is exactly the same as the original function. The exact value of this is represented by the letter e.

For all real values of x:
- **If $f(x) = e^x$ then $f'(x) = e^x$**
- **If $y = e^x$ then $\frac{dy}{dx} = e^x$**

A similar result holds for functions such as e^{5x}, e^{-x} and $e^{\frac{1}{2}x}$.

For all real values of x and for any constant k:
- **If $f(x) = e^{kx}$ then $f'(x) = ke^{kx}$**
- **If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$**

Example 1: Differentiate with respect to x.

a. e^{4x} b. $e^{-\frac{1}{2}x}$ c. $3e^{2x}$

a. $y = e^{4x}$
$\frac{dy}{dx} = 4e^{4x}$
Use the rule for differentiating e^{kx} with $k = 4$

b. $y = e^{-\frac{1}{2}x}$
$\frac{dy}{dx} = -\frac{1}{2}e^{-\frac{1}{2}x}$

c. $y = 3e^{2x}$
$\frac{dy}{dx} = 2 \times 3e^{2x} = 6e^{2x}$
To differentiate ae^{kx}, multiply the whole function by k. The derivate is kae^{kx}.

Exponential modelling

e^x can be used to model situations such as population growth, where the rate of increase is proportional to the size of the population at any given moment. Similarly, e^{-x} can be used to model radioactive decay, where the rate of decrease is proportional to the number of atoms remaining.

Example 2:
The density of a pesticide in a given section of field, P mg/m², can be modelled by the equation

$$P = 160e^{-0.006t}$$

where t is the time in days since the pesticide was first applied.

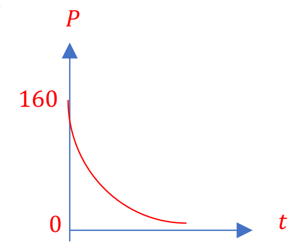
a. Use this model to estimate the density of pesticide after 15 days.
After 15 days, $t = 15$.
$P = 160e^{-0.006 \times 15}$
$P = 146.2$ mg/m²

b. Interpret the meaning of the value 160 in this model.
When $t = 0$, $P = 160e^0 = 160$, so 160 mg/m² is the initial density of pesticide in the field.

c. Show that $\frac{dP}{dt} = kP$, where k is a constant, and state the value of k.
$P = 160e^{-0.006t}$
$\frac{dP}{dt} = -0.96e^{-0.006t}$, so $k = -0.96$
If $y = e^{kx}$ then $\frac{dy}{dx} = ke^{kx}$

d. Interpret the significance of the sign of your answer to part c.
As k is negative, the density of the pesticide is decreasing (there is exponential decay)

e. Sketch the graph of P against t.



Logarithms

The inverses of exponential functions are called logarithms.
- $\log_a n = x$ is equivalent to $a^x = n$ $(a \neq 1)$

Example 3: Write each statement as a logarithm.

a. $3^2 = 9$ b. $2^7 = 128$ c. $64^{\frac{1}{2}} = 8$

a. $3^2 = 9$, so $\log_3 9 = 2$
b. $2^7 = 128$, so $\log_2 128 = 7$
c. $64^{\frac{1}{2}} = 8$, so $\log_{64} 8 = \frac{1}{2}$
Logarithms can take fractional or negative values

Laws of logarithms

Expressions involving more than one logarithm can be rearranged or simplified.
The laws of logarithms:
- $\log_a x + \log_a y = \log_a xy$ (the multiplication law)
- $\log_a x - \log_a y = \log_a \frac{x}{y}$ (the division law)
- $\log_a (x^k) = k\log_a x$ (the power law)

You should also recognise the following special cases:
- $\log_a \frac{1}{x} = \log_a (x^{-1}) = -\log_a x$ (the power law when $k = -1$)
- $\log_a a = 1$ $(a > 0, a \neq 1)$
- $\log_a 1 = 0$ $(a > 0, a \neq 1)$

Example 4: Write as a single logarithm.

a. $\log_3 6 + \log_3 7$
$= \log_3 (6 \times 7)$
$= 42$

b. $\log_2 15 - \log_2 3$
$= \log_2 (15 \div 3)$
$= \log_2 5$

c. $2\log_5 3 + 3\log_5 2$
$2\log_5 3 = \log_5 (3^2) = \log_5 9$
$3\log_5 2 = \log_5 (2^3) = \log_5 8$
$\log_5 9 + \log_5 8 = \log_5 72$

d. $\log_{10} 3 - 4\log_{10}\left(\frac{1}{2}\right)$
$4\log_{10}\left(\frac{1}{2}\right) = \log_{10}\left(\frac{1}{2}\right)^4 = \log_{10}\left(\frac{1}{16}\right)$
$\log_{10} 3 - \log_{10}\left(\frac{1}{16}\right) = \log_{10}\left(3 \div \frac{1}{16}\right) = \log_{10} 48$

Solving equations using logarithms

You can use logarithms and your calculator to solve equations of the form $a^x = b$. You can also solve more complicated equations by 'taking logs' of both sides.
- Whenever $f(x) = g(x)$, $\log_a f(x) = \log_a g(x)$

Example 5: Solve the following equations, giving your answers to 3 decimal places.

a. $3^x = 20$
So $x = \log_3 20 = 2.727$
Use the log button on your calculator

b. $5^{4x-1} = 61$
So $4x - 1 = \log_5 61$
$4x = \log_5 61 + 1$
$x = \frac{\log_5 61 + 1}{4} = 0.889$

Working with natural logarithms

- The graph of $y = \ln x$ is a reflection of the graph $y = e^x$ in the line $y = x$.

The graph of $y = \ln x$ passes through (1,0) and does not cross the y-axis.
The y-axis is an asymptote of the graph $y = \ln x$. This means that $\ln x$ is only defined for positive values of x.
Logarithms are the inverses of exponential functions. This rule can be used to solve equations involving powers and logarithms.
- $e^{\ln x} = \ln(e^x) = x$
- $\ln x = \log_e x$

Example 6: Solve these equations, giving your answers in exact form.

a. $e^x = 5$
When $e^x = 5$
$\ln(e^x) = \ln 5$
$x = \ln 5$
You can write the natural logarithm on both sides

b. $\ln x = 3$
When $\ln x = 3$
$e^{\ln x} = e^3$
$x = e^3$

Logarithms and non-linear data

Logarithms can also be used to manage and explore non-linear trends in data.

If $y = ax^n$ then the graph of $\log y$ against $\log x$ will be a straight line with gradient n and vertical intercept $\log a$.