Exponential Growth Questions

4 On 1 January 1900, a sculpture was valued at £80.

When the sculpture was sold on 1 January 1956, its value was £5000.

The value, £V, of the sculpture is modelled by the formula $V = Ak^t$, where t is the time in years since 1 January 1900 and A and k are constants.

(a) Write down the value of A. *(1 mark)*

(b) Show that $k \approx 1.07664$. *(3 marks)*

(c) Use this model to:

(i) show that the value of the sculpture on 1 January 2006 will be greater than £200 000; *(2 marks)*

(ii) find the year in which the value of the sculpture will first exceed £800 000.

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8 A disease is spreading through a colony of rabbits. There are 5000 rabbits in the colony. At time t hours, x is the number of rabbits infected. The rate of increase of the number of rabbits infected is proportional to the product of the number of rabbits infected and the number not yet infected.

(a) (i) Formulate a differential equation for $\frac{dx}{dt}$ in terms of the variables x and t and a constant of proportionality k. *(2 marks)*

(ii) Initially, 1000 rabbits are infected and the disease is spreading at a rate of 200 rabbits per hour. Find the value of the constant k.

(You are **not** required to solve your differential equation.) *(2 marks)*

(b) The solution of the differential equation in this model is

$$t = 4\ln\left(\frac{4x}{5000 - x}\right)$$

(i) Find the time after which 2500 rabbits will be infected, giving your answer in hours to one decimal place. *(2 marks)*

(ii) Find, according to this model, the number of rabbits infected after 30 hours. *(4 marks)*

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8 (a) (i) Solve the differential equation $\dfrac{dy}{dt} = y \sin t$ to obtain y in terms of t. *(4 marks)*

(ii) Given that $y = 50$ when $t = \pi$, show that $y = 50e^{-(1+\cos t)}$. *(3 marks)*

(b) A wave machine at a leisure pool produces waves. The height of the water, y cm, above a fixed point at time t seconds is given by the differential equation

$$\frac{dy}{dt} = y \sin t$$

(i) Given that this height is 50 cm after π seconds, find, to the nearest centimetre, the height of the water after 6 seconds. *(2 marks)*

(ii) Find $\dfrac{d^2y}{dt^2}$ and hence verify that the water reaches a maximum height after π seconds. *(4 marks)*

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4 A biologist is researching the growth of a certain species of hamster. She proposes that the length, x cm, of a hamster t days after its birth is given by

$$x = 15 - 12e^{-\frac{t}{14}}$$

(a) Use this model to find:

(i) the length of a hamster when it is born; *(1 mark)*

(ii) the length of a hamster after 14 days, giving your answer to three significant figures. *(2 marks)*

(b) (i) Show that the time for a hamster to grow to 10 cm in length is given by $t = 14 \ln\left(\dfrac{a}{b}\right)$, where a and b are integers. *(3 marks)*

(ii) Find this time to the nearest day. *(1 mark)*

(c) (i) Show that

$$\frac{dx}{dt} = \frac{1}{14}(15 - x)$$ *(3 marks)*

(ii) Find the rate of growth of the hamster, in cm per day, when its length is 8 cm. *(1 mark)*

Exponential Growth Answers

4(a)	$A = 80$	B1	1	
(b)	$5000 = 80 \times k^{56}$	M1		SC1 Verification. Need 62.51 or better
	$k = \sqrt[56]{\frac{5000}{80}} \approx 1.07664$	M1A1	3	**Or** using logs: M1 $\ln\left(\frac{5000}{80}\right) = 56 \ln k$ A1 $k = e^{\ln\left(\frac{62.5}{56}\right)}$ **Or** 3/3 for $k = 1.076636$ **Or** 1.076637 seen
(c)(i)	$V = 80 \times k^{106} = 200707$	M1A1	2	200648 using full register k
(ii)	$\ln 10000 = \ln k^t$	M1		
	$t = \frac{\ln 10000}{\ln k} = 124.7 \Rightarrow 2024$	M1A1	3	M1 $t \ln k = \ln 10000$ A1 CAO **Or** trial and improvement M1expression M1 125, 124, A1 2024
	Total		**9**	

8(a)(i)	$(5000 - x)$ seen in a product	B1		Could be implied, eg $5000a - xa$
	$\frac{dx}{dt} = kx(5000 - x)$	B1	2	
(ii)	$200 = k \times 1000 \times (5000 - 1000)$	M1		$\frac{dx}{dt} = 200, x = 1000$ in their diff. equation Condone ts and $t = 0$ for M1
	$k = 0.00005$	A1	2	CAO OE
(b)(i)	$t = 4\ln\left(\frac{4 \times 2500}{5000 - 2500}\right) = 5.5$ (hours)	M1 A1	2	$x \to 2500$ (or $4 \ln 4$) CAO
(ii)	$e^{\frac{30}{4}}$	B1		
	$e^{7.5} = \frac{4x}{5000 - x}$	M1		OE
	$5000 \times e^{7.5} = x\left(4 + e^{7.5}\right)$	m1		Soluble for x
	$x = 4988.96.. \Rightarrow 4989$ rabbits infected	A1	4	Or 4988 or 4990; integer value only
	Total		**10**	

8(a)(i)	$\int \frac{\mathrm{d}y}{y} = \int \sin t \,\mathrm{d}t$	M1		Attempt to separate and integrate
	$\ln y = -\cos t + C$	A1,A1		A1 for $\ln y$; A1 for $-\cos t$; condone missing C
	$y = Ae^{-\cos t}$	A1	4	A present; or $y = e^{-\cos t + C}$
(ii)	$y = 50,\ t = \pi:\quad 50 = Ae^{-\cos\pi} = Ae$	M1 A1		Substitute $y = 50,\ t = \pi$ to find constant Can have $50 = e^{1+C}$ if substituted in above $e^C = \frac{50}{e}$
	$y = 50e^{-1}e^{-\cos t}$	A1	3	AG (convincingly obtained)
	Alternative: Must have a constant in answer to (a)(i) $y = Ae^{-\cos t}$ or $y = e^{-\cos t + c}$ or $\ln y = -\cos t + c$			**Alternative:** Substitute $y = 50,\ t = \pi$ into $\ln y = -\cos t + c$ M1 $\ln y = -\cos t + \ln 50 - 1$ A1
	$50 = Ae^{-\cos\pi}$ $\quad 50 = e^{-\cos\pi + c}$ $\quad \ln 50 = -\cos\pi + c$	(M1)		$\ln\frac{y}{50} = -1 - \cos t$ (AG) A1
	$50 = Ae$ $\quad 50 = e^{1+c}$ $\quad \ln y = -\cos t + \ln 50 - 1$	(A1)		
	$y = 50e^{-1-\cos t}$ $\quad y = e^{-\cos t}\frac{50}{e}$ $\quad \ln\left(\frac{y}{50}\right) = -1 - \cos t$	(A1)		
(b)(i)	$t = 6:\ \ y = 50e^{-1}e^{-\cos 6} = 7.0417... \approx 7\text{cm}$	M1A1	2	Degrees 6.8 SC1 7 or 7.0 for A1
(ii)	$t = \pi \Rightarrow\ (\sin t = 0 \Rightarrow)\ \frac{\mathrm{d}y}{\mathrm{d}t} = 0$	B1		Condone x for t
	$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = y\cos t + \frac{\mathrm{d}y}{\mathrm{d}t}\sin t$	M1		For attempt at product rule including $\frac{\mathrm{d}y}{\mathrm{d}t}$ term; must have $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} =$
		A1		
	$t = \pi$ $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = y\cos\pi + \frac{\mathrm{d}y}{\mathrm{d}t}\sin\pi$ $= -50 \Rightarrow$ max	A1	4	Accept $= -y$, with explanation that y is never negative

8(b)(ii) (cont)	**Alternative:** $y = 50e^{-(1+\cos t)} = \frac{50}{e}e^{-\cos t}$			
	$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{50}{e}e^{-\cos t} \times \sin t = 0$ at $t = \pi$	(B1)		
	$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{50}{e}e^{-\cos t} \times \cos t + \frac{50}{e}e^{-\cos t} \times \sin^2 t$	(M1) (A1)		Attempt at product rule Correct
	Substitute $t = \pi \to -50 \Rightarrow$ max	(A1)		
	Total		13	

4(a)(i)	$t=0:\ x=3$	B1	1	
(ii)	$t=14:\ x=15-12\mathrm{e}^{-1}$	M1		or $15-12\,\mathrm{e}^{\frac{-14}{14}}$
	$=10.6$	A1	2	CAO
(b)(i)	$-5=-12\mathrm{e}^{-\frac{t}{14}}$	M1		substitute $x=10$; rearrange to form $p=q\mathrm{e}^{-\frac{t}{14}}$
	$\ln\left(\frac{5}{12}\right)=-\frac{t}{14}$ (OE)	m1		take lns correctly
	$t=14\ln\left(\frac{12}{5}\right)$	A1	3	must come from correct working
(ii)	$t=12.256\ldots\approx 12$ days	B1F	1	ft on a, b if $a > b$; accept $t = 12$ NMS Accept 12 from incorrect working in b(i) Accept 13 if 12.2 or 12.3 seen
(c)(i)	$\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{14}\times -12\mathrm{e}^{-\frac{t}{14}}$	M1		differentiate; allow sign error condone $\frac{\mathrm{d}y}{\mathrm{d}x}$ used consistently
	$=-\frac{1}{14}(x-15)$	m1		Or $\frac{1}{14}\left(12\mathrm{e}^{-\frac{t}{14}}\right)$ and $12\mathrm{e}^{-\frac{t}{14}}=15-x$ seen
	$=\frac{1}{14}(15-x)$	A1	3	AG – be convinced CSO
	Alt: $t=-14\ln\left(\frac{15-x}{12}\right)$	(M1)		attempt to solve given equation for t
	$\frac{\mathrm{d}t}{\mathrm{d}x}=\dfrac{-14\left(-\frac{1}{12}\right)}{\left(\frac{15-x}{12}\right)}$	(m1)		differentiate wrt x, with $\dfrac{1}{\frac{15-x}{12}}$ seen; OE
	$\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{14}{15-x}\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{14}(15-x)$	(A1)	(3)	AG – be convinced
	Alt: (backwards) $\int\frac{\mathrm{d}x}{15-x}=\int\frac{\mathrm{d}t}{14}=\pm 14\ln(15-x)=t+c$	(M1)		
	Use $(0,3)$: $-14\ln(15-x)+14\ln 12=t$	(m1)		
	Solve for x: $x=15-12\mathrm{e}^{-\frac{t}{14}}$	(A1)	(3)	All steps shown
(ii)	rate of growth = 0.5 (cm per day)	B1	1	Accept $\frac{7}{14}$
	Total		**11**	