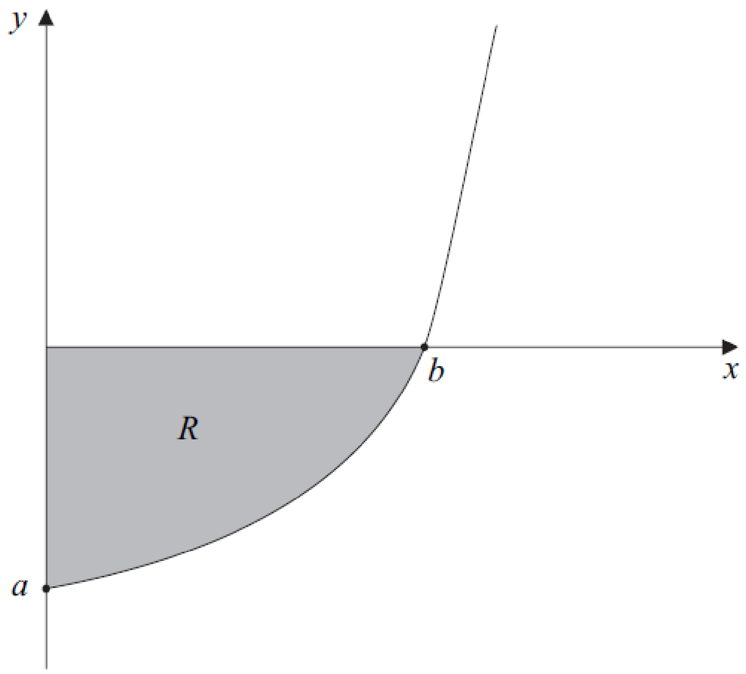
Exponentials & Logarithms Questions

5 The diagram shows part of the graph of $y = e^{2x} - 9$. The graph cuts the coordinate axes at $(0, a)$ and $(b, 0)$.



(a) State the value of a, and show that $b = \ln 3$. *(3 marks)*

(b) Show that $y^2 = e^{4x} - 18e^{2x} + 81$. *(1 mark)*

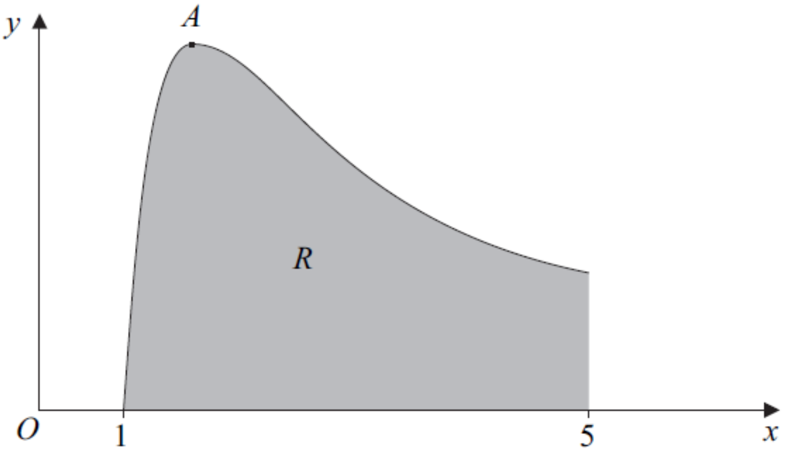
(c) The shaded region R is rotated through 360° about the x-axis. Find the volume of the solid formed, giving your answer in the form $\pi(p \ln 3 + q)$, where p and q are integers. *(6 marks)*

(d) Sketch the curve with equation $y = |e^{2x} - 9|$ for $x \geqslant 0$. *(2 marks)*

---

9 (a) Given that $y = x^{-2} \ln x$, show that $\dfrac{dy}{dx} = \dfrac{1 - 2\ln x}{x^3}$. *(4 marks)*

(c) The sketch shows the graph of $y = x^{-2} \ln x$.



(i) Using the answer to part (a), find, in terms of e, the x-coordinate of the stationary point A. *(2 marks)*

---

5 (a) A curve has equation $y = e^{2x} - 10e^{x} + 12x$.

(i) Find $\frac{dy}{dx}$. *(2 marks)*

(ii) Find $\frac{d^2y}{dx^2}$. *(1 mark)*

(b) The points P and Q are the stationary points of the curve.

(i) Show that the x-coordinates of P and Q are given by the solutions of the equation

$$e^{2x} - 5e^{x} + 6 = 0$$ *(1 mark)*

(ii) By using the substitution $z = e^x$, or otherwise, show that the x-coordinates of P and Q are $\ln 2$ and $\ln 3$. *(3 marks)*

(iii) Find the y-coordinates of P and Q, giving each of your answers in the form $m + 12 \ln n$, where m and n are integers. *(3 marks)*

(iv) Using the answer to part (a)(ii), determine the nature of each stationary point.

(b) (i) Given that $y = x \ln x$, find $\dfrac{dy}{dx}$. *(2 marks)*

(ii) Hence, or otherwise, find $\displaystyle\int \ln x \, dx$. *(2 marks)*

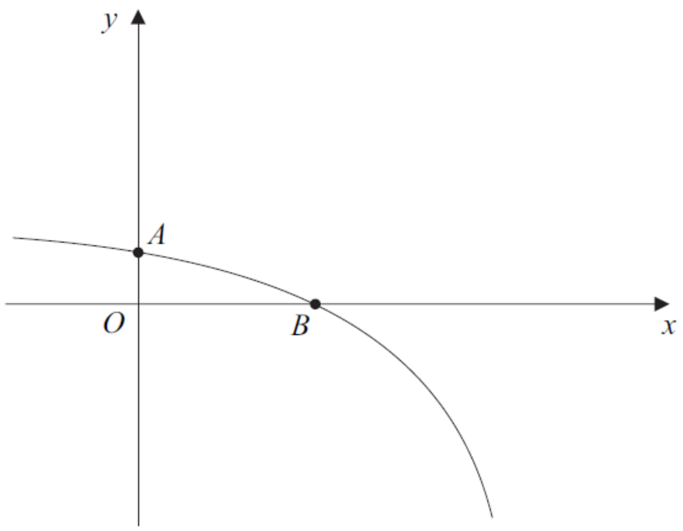
(iii) Find the exact value of $\displaystyle\int_1^5 \ln x \, dx$. *(2 marks)*

(b) (i) Find $\dfrac{dx}{dy}$ when $x = 2y^3 + \ln y$. *(1 mark)*

(ii) Hence find an equation of the tangent to the curve $x = 2y^3 + \ln y$ at the point $(2,1)$. *(3 marks)*

---

9 The sketch shows the graph of $y = 4 - e^{2x}$. The curve crosses the y-axis at the point A and the x-axis at the point B.



(a) (i) Find $\int (4 - e^{2x})dx$. *(2 marks)*

(ii) Hence show that $\int_0^{\ln 2} (4 - e^{2x})dx = 4\ln 2 - \frac{3}{2}$. *(2 marks)*

(b) (i) Write down the y-coordinate of A. *(1 mark)*

(ii) Show that $x = \ln 2$ at B. *(2 marks)*

(c) Find the equation of the normal to the curve $y = 4 - e^{2x}$ at the point B. *(4 marks)*

(d) Find the area of the region enclosed by the curve $y = 4 - e^{2x}$, the normal to the curve at B and the y-axis. *(3 marks)*

---

1 (a) Differentiate $\ln x$ with respect to x. *(1 mark)*

(b) Given that $y = (x+1)\ln x$, find $\frac{dy}{dx}$. *(2 marks)*

(c) Find an equation of the normal to the curve $y = (x+1)\ln x$ at the point where $x = 1$.

---

7 (a) A curve has equation $y = (x^2 - 3)e^x$.

(i) Find $\frac{dy}{dx}$. *(2 marks)*

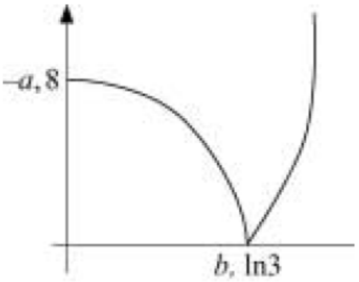
(ii) Find $\frac{d^2y}{dx^2}$. *(2 marks)*

(b) (i) Find the x-coordinate of each of the stationary points of the curve. *(4 marks)*

(ii) Using your answer to part (a)(ii), determine the nature of each of the stationary points. *(2 marks)*

Exponentials & Logarithms Answers

5(a)	$a = -8$	B1		
	$e^{2x} - 9 = 0$	M1		
	$e^{2x} = 9$			
	$2x = \ln 9$			
	$x = \ln 3$	A1	3	AG Condone verification
(b)	$\left(e^{2x} - 9\right)^2 = e^{4x} - 18e^{2x} + 81$	B1	1	AG
(c)	$V = \pi \int y^2 \, (dx)$	B1		
	$= (\pi) \int e^{4x} - 18e^{2x} + 81 \ dx$	M1		
	$= (\pi) \left[\frac{e^{4x}}{4} - 9e^{2x} + 81x \right]_0^{\ln 3}$	M1 A1		1ST or 2nd term correct All correct

	$= (\pi) \left[\frac{e^{4x}}{4} - 9e^{2x} + 81x \right]_0^{\ln 3}$	M1 A1		1ST or 2nd term correct All correct
	$= (\pi) \left[\left(\frac{e^{\ln 81}}{4} - 9e^{\ln 9} + 81 \ln 3 \right) - \left(\frac{1}{4} - 9 \right) \right]$	m1		Attempt at limits with ln3
	$= \pi [81 \ln 3 - 52]$	A1	6	
(d)		M1		Modulus graph
	$-a$	A1F	2	All correct
	Total		**12**	

9(a)	$y = x^{-2} \ln x$			
	$\frac{dy}{dx} = x^{-2} \frac{1}{x} - 2x^{-3} \ln x$	M1 A1 A1		Use of product or quotient each term
	$= \frac{1 - 2 \ln x}{x^3}$	A1	4	Convincing argument $x^{-2} \times \frac{1}{x} = x^{-3}$ AG

(c)(i)	At A, $\frac{dy}{dx}=0$ $1-2\ln x=0$ $\ln x=\frac{1}{2}$ $x=e^{\frac{1}{2}}$	M1 A1	2	Attempt at $\ln x = k$

5(a)	$y = e^{2x} - 10e^{x} + 12x$			
(i)	$\frac{dy}{dx}=2e^{2x}-10e^{x}+12$	B1 B1	2	$2e^{2x}$ remaining terms correct, no extras
(ii)	$\frac{d^2y}{dx^2}=4e^{2x}-10e^{x}$	B1F	1	ft 1 slip
(b)(i)	$2e^{2x}-10e^{x}+12=0$ $e^{2x}-5e^{x}+6=0$	B1	1	AG (be convinced)
(ii)	$z^2-5z+6=0$	M1		use of $z = e^{x}$ oe
	$z=2,3$ $z=2,\ e^{x}=2$ $x=\ln 2$	M1		finding $e^{x} =$ their 2,3
	$z=3,\ e^{x}=3$ $x=\ln 3$	A1	3	all correct AG SC: verification
				$\ln 2$ (B1) $\ln 3$ (B1)
(iii)	$x=\ln 2$: $y=e^{2\ln 2}-10e^{\ln 2}+12\ln 2$ or $2^2-10\times 2+12\ln 2$ $= 4 - 20 + 12\ln 2$	M1		either substitution of their $x = \ln 2$ $(e^{x}=2)$ or their $x = \ln 3$ $(e^{x}=3)$
	$= -16 + 12\ln 2$	A1		
	$x=\ln 3$: $y=e^{2\ln 3}-10e^{\ln 3}+12\ln 3$ $= 9 - 30 + 12\ln 3$ $= -21 + 12\ln 3$	A1	3	
(iv)	$x=\ln 2$: $\frac{d^2y}{dx^2}=4e^{2\ln 2}-10e^{\ln 2}$	M1		use of; in either of their $e^{x}=2,3$ into their $\frac{d^2y}{dx^2}$

	$= 16 - 20 = -4$			
	$\therefore$ maximum	A1		CSO
	$x = \ln 3$:			
	$\frac{d^2y}{dx^2} = 4e^{2\ln 3} - 10e^{\ln 3}$			
	$= 36 - 30 = 6$			
	$\therefore$ minimum	A1	3	CSO
	Total		**13**	

(b)(i)	$y = x \ln x$			
	$\frac{dy}{dx} = x \times \frac{1}{x} + \ln x$	M1		use of product rule (only differentiating, 2 terms with + sign)
	$= \ln x + 1$	A1	2	
(ii)	$\int (\ln x + 1) dx = x \ln x$	M1		OE; attempt at parts with $u = \ln x$
	$\int \ln x \, dx = x \ln x - x (+c)$	A1	2	
(iii)	$\int_1^5 \ln x \, dx = [x \ln x - x]_1^5$			
	$= (5 \ln 5 - 5) - (1 \ln 1 - 1)$	M1		correct substitution of limits into their (ii) provided $\ln x$ is involved
	$5 \ln 5 - 4$	A1	2	ISW
	Total		**9**	

(b)(i)	$x = 2y^3 + \ln y$			
	$\frac{dx}{dy} = 6y^2 + \frac{1}{y}$	B1	1	
(ii)	At $(2,1)$			
	$\frac{dx}{dy} = 6 + 1 = 7$	M1		
	$\frac{dy}{dx} = \frac{1}{7}$	A1√		May be implied
	$(y - 1) = \frac{1}{7}(x - 2)$	A1	3	OE

9(a)(i)	$\int\left(4-e^{2x}\right)dx$			
	$=4x-\frac{1}{2}e^{2x}\ (+c)$	B1 B1	2	$4x$ $-\frac{1}{2}e^{2x}$
(ii)	$\int_0^{\ln 2}=\left[4x-\frac{1}{2}e^{2x}\right]_0^{\ln 2}$			
	$=\left[4\ln 2-\frac{1}{2}e^{2\ln 2}\right]-\left[(0)-\frac{1}{2}(e^0)\right]$	M1		Substitute both ln 2 and 0 correctly into an integrated expression
	$=4\ln 2-2+\frac{1}{2}$			Convincing
	$=4\ln 2-\frac{3}{2}$	A1	2	**AG**
(b)(i)	$x=0$ $y=4-1=\underline{\underline{3}}$	B1	1	
(ii)	At B, $y=0$ $4-e^{2x}=0$ $e^{2x}=4$	M1		Or reverse argument
	$x=\ln 2$	A1	2	**AG**
(c)	$\frac{dy}{dx}=-2e^{2x}$	B1		
	$x=\ln 2$, Gradient $=-2e^{2\ln 2}$ $=-8$	M1		$x=\ln 2$ into ke^{2x}
	Gradient normal $=\frac{1}{8}=\frac{1}{2e^{2\ln 2}}$	A1		OE
	Equation $y=\frac{1}{8}x-\frac{1}{8}\ln 2$	A1	4	OE
(d)	When $x=0$ $y=-\frac{1}{8}\ln 2$	M1		Attempt to integrate their line and substitute $x=0, \ln 2$
	Area Δ $=\frac{1}{16}(\ln 2)^2$ condone – ve sign $=0.03$	A1√		$\frac{1}{2}(\text{their } y)\times\ln 2$
	Total area $=4\ln 2-\frac{3}{2}+\frac{1}{16}(\ln 2)^2=1.30$ AWRT	A1	3	CSO
	Total		**14**	

1(a)	$y = \ln x$ $\frac{dy}{dx} = \frac{1}{x}$	B1	1	penalise + c once on 1(a) or 2(a)
(b)	$y = (x+1)\ln x$ $\frac{dy}{dx} = (x+1) \times \frac{1}{x} + \ln x$	M1 A1	2	product rule
(c)	$y = (x+1)\ln x$ $\frac{dy}{dx} = \frac{1}{x} + 1 + \ln x$			
	$x = 1$: $\frac{dy}{dx} = 1 + 1 = 2$	M1		substitute $x = 1$ into their $\frac{dy}{dx}$
	Grad normal $= -\frac{1}{2}$	M1 A1		use of $m_1 m_2 = -1$ CSO
	$y = -\frac{1}{2}(x-1)$	A1	4	OE
	Total		**7**	

7(a)(i)	$y = (x^2 - 3)e^x$ $\frac{dy}{dx} = (x^2 - 3)e^x + 2xe^x$	M1 A1	2	product rule
(ii)	$\frac{d^2y}{dx^2} = (x^2 - 3)e^x + 2xe^x + 2xe^x + 2e^x$	M1 A1	2	product rule from their $\frac{dy}{dx}$
(b)(i)	$\frac{dy}{dx} = 0$ $\Rightarrow e^x(x^2 + 2x - 3) = 0$	M1		$e^x f(x) = 0$ from $\frac{dy}{dx} = 0$
	$e^x(x+3)(x-1) = 0$	m1		attempt at factorising or use of formula
	$\therefore x = -3, 1$	A1 A1	4	first correct solution second correct solution, and no others SC No working shown: $x = -3$ B2, $x = 1$ B2
(ii)	$x = -3$ $y'' = -4e^x$ max (-0.2)	M1		Condone slip
	$x = 1$ $y'' = 4e^x$ min (10.9)	A1	2	
	Total		**10**	