| **DIFFERENTIATION** | *Answers* |

1   **a** $\dfrac{dy}{dx} = -\dfrac{1}{4}x^{-2} - \dfrac{1}{x}$

   $x = 1$ $\therefore$ grad $= -\dfrac{5}{4}$

   b grad of normal $= \dfrac{4}{5}$

   $\therefore\ y - \dfrac{1}{4} = \dfrac{4}{5}(x - 1)$

   $16x - 20y - 11 = 0$

2   **a** $\dfrac{dy}{dx} = 1 \times e^{-2x} + x \times (-2e^{-2x})$

   $= e^{-2x}(1 - 2x)$

   $\dfrac{d^2y}{dx^2} = -2e^{-2x} \times (1 - 2x) + e^{-2x} \times (-2)$

   $= 4e^{-2x}(x - 1)$

   b SP: $e^{-2x}(1 - 2x) = 0$

   $x = \dfrac{1}{2}$

   $\therefore\ (\dfrac{1}{2},\ \dfrac{1}{2}e^{-1})$

   when $x = \dfrac{1}{2}$, $\dfrac{d^2y}{dx^2} = -2e^{-1}$

   $\dfrac{d^2y}{dx^2} < 0$ $\therefore$ maximum

3   **a** $y = 0 \Rightarrow x = \sqrt{3}$

   $\therefore\ (\sqrt{3},\ 0)$

   b $= \dfrac{1}{2}(e^y + 2)^{-\frac{1}{2}} \times e^y$

   $= \dfrac{e^y}{2\sqrt{e^y + 2}}$

   c $\dfrac{dy}{dx} = 1 \div \dfrac{dx}{dy} = \dfrac{2\sqrt{e^y + 2}}{e^y}$

   grad $= 2\sqrt{3}$

   $\therefore\ y - 0 = 2\sqrt{3}\,(x - \sqrt{3})$

   at Q, $x = 0$ $\therefore\ y = -6$
   area $= \dfrac{1}{2} \times \sqrt{3} \times 6 = 3\sqrt{3}$

4   **a** $t = 0, m = 680$

   $t = 100, m = 653.63$

   % red'n $= \dfrac{680 - 653.63}{680} \times 100\% = 3.88\%$ (3sf)

   b $640 = 600 + 80e^{-0.004t}$

   $t = \dfrac{-1}{0.004}\ln \dfrac{1}{2} = 173$ (3sf)

   c $\dfrac{dm}{dt} = 80 \times (-0.004)e^{-0.004t} = -0.32e^{-0.004t}$

   $t = 150$, $\dfrac{dm}{dt} = -0.176$

   $\therefore$ mass decreasing at 0.176 g yr^{-1} (3sf)

5   **a** $= \dfrac{1}{2}(\sin x + \cos x)^{-\frac{1}{2}} \times (\cos x - \sin x)$

   $= \dfrac{\cos x - \sin x}{2\sqrt{\sin x + \cos x}}$

   b $= \dfrac{d}{dx}[\ln(x - 1) - \ln(2x + 1)]$

   $= \dfrac{1}{x-1} - \dfrac{1}{2x+1} \times 2$

   $= \dfrac{1}{x-1} - \dfrac{2}{2x+1}$

6   **a** $\dfrac{dy}{dx} = 5(2x - 3)^4 \times 2 = 10(2x - 3)^4$

   $x = 1$ $\therefore$ grad $= 10$

   $\therefore\ y + 1 = 10(x - 1)$

   $[\,y = 10x - 11\,]$

   b at Q $10(2x - 3)^4 = 10$

   $2x - 3 = \pm 1$
   $x = 1$ (at P) or 2
   $\therefore\ Q\,(2,\ 1)$

7   **a** $\dfrac{dy}{dx} = -2(x^2 - 5)^{-2} \times 2x = \dfrac{-4x}{(x^2-5)^2}$

SP: $\dfrac{-4x}{(x^2-5)^2} = 0$

$x = 0$

$\therefore \ (0, -\tfrac{2}{5})$

b $x = 3, \ y = \tfrac{1}{2}$

grad $= -\tfrac{3}{4}$

$\therefore \ y - \tfrac{1}{2} = -\tfrac{3}{4}(x - 3)$

$4y - 2 = -3x + 9$

$3x + 4y - 11 = 0$

8   **a**

b $y = ae^x + a$

swap $x = ae^y + a$

$y = \ln \dfrac{x-a}{a}$

$f^{-1} : x \to \ln \dfrac{x-a}{a}, \ x \in \mathbb{R}, \ x > a$

c $x = 1 \ \therefore \ y = ae + a$

$f'(x) = ae^x, \ \text{grad} = ae$

$\therefore \ y - (ae + a) = ae(x - 1)$

$[\, y = aex + a \,]$

9   **a** $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right)$

$= \dfrac{-\sin x \times \sin x - \cos x \times \cos x}{\sin^2 x}$

$= -\dfrac{\sin^2 x + \cos^2 x}{\sin^2 x}$

$= -\dfrac{1}{\sin^2 x} = -\text{cosec}^2 x$

b $\dfrac{dy}{dx} = e^x \times \cot x + e^x \times (-\text{cosec}^2 x)$

$= e^x(\cot x - \text{cosec}^2 x)$

SP: $e^x(\cot x - \text{cosec}^2 x) = 0$

$e^x \neq 0 \ \therefore \ \cot x = \text{cosec}^2 x$
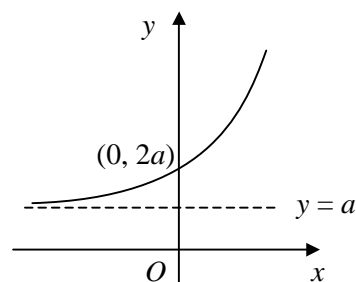
$\dfrac{\cos x}{\sin x} = \dfrac{1}{\sin^2 x}$

$\sin x \cos x = 1$

$\sin 2x = 2$

$|\sin 2x| \leq 1 \ \therefore \ \text{no solutions}$

$\therefore \ \text{no turning points}$

10   **a** $\dfrac{dy}{dx} = 3(2 + \ln x)^2 \times \dfrac{1}{x}$

$= \dfrac{3}{x}(2 + \ln x)^2$

b   SP: $\dfrac{3}{x}(2 + \ln x)^2 = 0$

$\ln x = -2$

$x = e^{-2}$

$\therefore \ (e^{-2}, 0)$

c $x = e, \ y = 27$

grad $= \dfrac{27}{e}$

$\therefore \ y - 27 = \dfrac{27}{e}(x - e)$

$y = \dfrac{27}{e} x$

when $x = 0, \ y = 0$

$\therefore \ \text{passes through origin}$

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11 **a** $= \dfrac{1}{9 - x^2} \times (-2x)$

$= \dfrac{-2x}{9 - x^2}$

b SP: $\dfrac{-2x}{9 - x^2} = 0$

$x = 0$

$\therefore (0, \ln 9)$

c $x = 1,\ y = \ln 8 = \ln 2^3 = 3 \ln 2$

grad $= -\frac{1}{4}$

$\therefore$ grad of normal $= 4$

$\therefore\ y - 3 \ln 2 = 4(x - 1)$

$y = 4x - 4 + 3 \ln 2$

12 **a** model A: $t = 3$, $M = 764$ (3sf)

model B: $t = 3$, $M = 732$ (3sf)

b model A:

$\dfrac{\mathrm{d}M}{\mathrm{d}t} = 1500(3t + 2)^{-2} \times 3 = \dfrac{4500}{(3t + 2)^2}$

$t = 3,\ \dfrac{\mathrm{d}M}{\mathrm{d}t} = 37.2$

$\therefore$ increasing at 37.2 tonnes yr^{-1} (3sf)

model B:

$\dfrac{\mathrm{d}M}{\mathrm{d}t} = 1500[2 + 5 \ln (t + 1)]^{-2} \times \dfrac{5}{t + 1}$

$= \dfrac{7500}{(t + 1)[2 + 5\ln(t + 1)]^2}$

$t = 3,\ \dfrac{\mathrm{d}M}{\mathrm{d}t} = 23.5$

$\therefore$ increasing at 23.5 tonnes yr^{-1} (3sf)