

## DIFFERENTIATION

## Answers

- 1**  $6x + 1 \times y + x \times \frac{dy}{dx} - 2y \frac{dy}{dx} = 0$   
 $6x + y = \frac{dy}{dx}(2y - x)$   
 $\frac{dy}{dx} = \frac{6x + y}{2y - x}$
- 2** **a**  $\frac{dx}{d\theta} = -a \sin \theta, \frac{dy}{d\theta} = a(\cos \theta - 1)$   
 $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a(\cos \theta - 1)}{-a \sin \theta} = \frac{1 - \cos \theta}{\sin \theta}$   
 $= \frac{1 - (1 - 2\sin^2 \frac{\theta}{2})}{2\sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \tan \frac{\theta}{2}$   
**b**  $x = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$   
 $\therefore y = a(1 - \frac{\pi}{2}), \text{ grad} = 1$   
 $\therefore y = x + a(1 - \frac{\pi}{2})$
- 3** **a**  $\frac{dx}{d\theta} = -\sin \theta, \frac{dy}{d\theta} = \cos 2\theta$   
 $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{\cos 2\theta}{-\sin \theta}$   
 $= -\operatorname{cosec} \theta \cos 2\theta$   
**b**  $x = 0 \Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$   
**c**  $\theta = \frac{\pi}{2}, \text{ grad} = -1 \times (-1) = 1$   
 $\theta = \frac{3\pi}{2}, \text{ grad} = 1 \times (-1) = -1$   
product of gradients  $= 1 \times (-1) = -1$   
 $\therefore$  tangents are perpendicular  
**d**  $y = \frac{1}{2} \sin 2\theta = \sin \theta \cos \theta$   
 $y^2 = \sin^2 \theta \cos^2 \theta = \cos^2 \theta (1 - \cos^2 \theta)$   
 $\therefore y^2 = x^2(1 - x^2)$
- 4** **a**  $2x - 4 \times y - 4x \times \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$   
 $2x - 4y = \frac{dy}{dx}(4x - 2y)$   
 $\frac{dy}{dx} = \frac{2x - 4y}{4x - 2y} = \frac{x - 2y}{2x - y}$   
**b** grad = 3  
 $\therefore y - 10 = 3(x - 2) \quad [y = 3x + 4]$   
**c**  $\frac{x - 2y}{2x - y} = 3$   
 $x - 2y = 3(2x - y)$   
 $y = 5x$ , sub. into eqn of curve  
 $x^2 - 4x(5x) + (5x)^2 = 24$   
 $x^2 = 4$   
 $x = 2$  (at P) or  $-2 \therefore (-2, -10)$
- 5** **a**  $\frac{dx}{dt} = 2t, \frac{dy}{dt} = 2t - 1$   
 $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{2t - 1}{2t}$   
 $\therefore \frac{2t - 1}{2t} = 0$   
 $t = \frac{1}{2}$   
 $\therefore (\frac{9}{4}, -\frac{1}{4})$   
**b**  $x = 3 \Rightarrow t^2 + 2 = 3 \Rightarrow t = \pm 1$   
 $y = 2 \Rightarrow t^2 - t = 2$   
 $t^2 - t - 2 = 0$   
 $(t - 2)(t + 1) = 0$   
 $t = -1$  or  $2$   
 $\therefore$  at  $(3, 2), t = -1$   
 $\therefore \text{grad} = \frac{3}{2}$   
 $\therefore y - 2 = \frac{3}{2}(x - 3)$   
 $2y - 4 = 3x - 9$   
 $3x - 2y = 5$
- 6**  $3x^2 - 3 + 1 \times y + x \times \frac{dy}{dx} - 4y \frac{dy}{dx} = 0$   
 $3x^2 - 3 + y = \frac{dy}{dx}(4y - x)$   
 $\frac{dy}{dx} = \frac{3x^2 - 3 + y}{4y - x}$   
grad =  $\frac{1}{3}$   
 $\therefore$  grad of normal =  $-3$   
 $\therefore y - 1 = -3(x - 1)$   
 $y = 4 - 3x$

- 7 a**  $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ ,  $\frac{dV}{dt} = 80$   
 $\frac{dV}{dh} = 40\pi \times 0.1e^{0.1h} = 4\pi e^{0.1h}$   
 $h = 4$ ,  $\frac{dV}{dh} = 4\pi e^{0.4}$   
 $\therefore 80 = 4\pi e^{0.4} \times \frac{dh}{dt}$ ,  $\frac{dh}{dt} = 4.27$   
 $\therefore$  depth increasing at  $4.27 \text{ cm s}^{-1}$  (3sf)
- b** after 5 seconds,  $V = 5 \times 80 = 400$   
 $\therefore 400 = 40\pi(e^{0.1h} - 1)$   
 $h = 10 \ln\left(\frac{10}{\pi} + 1\right) = 14.31$   
 $\therefore \frac{dV}{dh} = 4\pi e^{1.431}$   
 $\therefore 80 = 4\pi e^{1.431} \times \frac{dh}{dt}$ ,  $\frac{dh}{dt} = 1.52$   
 $\therefore$  depth increasing at  $1.52 \text{ cm s}^{-1}$  (3sf)

- 8 a**  $\frac{dx}{dt} = \frac{1 \times (1+t) - t \times 1}{(1+t)^2} = \frac{1}{(1+t)^2}$   
 $\frac{dy}{dt} = \frac{1 \times (1-t) - t \times (-1)}{(1-t)^2} = \frac{1}{(1-t)^2}$   
 $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{1}{(1-t)^2} \div \frac{1}{(1+t)^2}$   
 $= \frac{(1+t)^2}{(1-t)^2} = \left(\frac{1+t}{1-t}\right)^2$
- b**  $t = \frac{1}{2} \therefore x = \frac{1}{3}$ ,  $y = 1$   
grad = 9  $\therefore$  grad of normal =  $-\frac{1}{9}$   
 $\therefore y - 1 = -\frac{1}{9}(x - \frac{1}{3})$   
 $27y - 27 = -3x + 1$   
 $3x + 27y = 28$
- c**  $\frac{3t}{1+t} + \frac{27t}{1-t} = 28$   
 $3t(1-t) + 27t(1+t) = 28(1-t^2)$   
 $26t^2 + 15t - 14 = 0$   
 $(13t + 14)(2t - 1) = 0$   
 $t = \frac{1}{2}$  (at  $P$ ) or  $-\frac{14}{13}$   
 $\therefore t = -\frac{14}{13}$  at  $Q$

- 9**  $2 + 2x \times y + x^2 \times \frac{dy}{dx} - 2y \frac{dy}{dx} = 0$   
 $2 + 2xy = \frac{dy}{dx}(2y - x^2)$   
 $\frac{dy}{dx} = \frac{2 + 2xy}{2y - x^2}$   
 $\therefore \frac{2 + 2xy}{2y - x^2} = 0$ ,  $2 + 2xy = 0$   
 $xy = -1$ ,  $y = -\frac{1}{x}$
- sub.  $2x + x^2(-\frac{1}{x}) - (-\frac{1}{x})^2 = 0$   
 $2x - x - \frac{1}{x^2} = 0$   
 $x = \frac{1}{x^2}$ ,  $x^3 = 1$   
 $x = 1 \therefore (1, -1)$

- 10 a**  $\frac{dx}{d\theta} = a \sec \theta \tan \theta$ ,  $\frac{dy}{d\theta} = 2a \sec^2 \theta$   
 $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{2a \sec^2 \theta}{a \sec \theta \tan \theta} = 2 \operatorname{cosec} \theta$
- b**  $\theta = \frac{\pi}{4}$ ,  $x = \sqrt{2} a$ ,  $y = 2a$   
grad =  $2\sqrt{2}$   
 $\therefore$  grad of normal =  $-\frac{1}{2\sqrt{2}}$   
 $\therefore y - 2a = -\frac{1}{2\sqrt{2}}(x - \sqrt{2} a)$   
 $2\sqrt{2} y - 4\sqrt{2} a = -x + \sqrt{2} a$   
 $x + 2\sqrt{2} y = 5\sqrt{2} a$
- c**  $y^2 = 4a^2 \tan^2 \theta = 4a^2(\sec^2 \theta - 1)$   
 $\sec \theta = \frac{x}{a}$   
 $\therefore y^2 = 4a^2[(\frac{x}{a})^2 - 1]$   
 $y^2 = 4(x^2 - a^2)$