**1.** The curve $C$ has parametric equations

$$x = \sin 2\theta \qquad y = \operatorname{cosec}^3\theta \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of $\theta$

**(3)**

(b) Hence find the exact value of the gradient of the tangent to $C$ at the point where $y = 8$

**(3)**

a) $y = \operatorname{cosec}^3\theta$

$\frac{dy}{d\theta} = 3\operatorname{cosec}^2\theta \times -\operatorname{cosec}\theta\cot\theta$

$= -3\operatorname{cosec}^3\theta\cot\theta$ (1)

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$ (1)

$x = \sin 2\theta$

$\frac{dx}{d\theta} = 2\cos 2\theta$

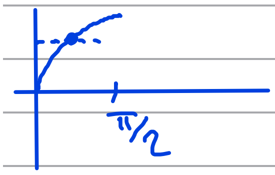
$\frac{dy}{dx} = \frac{-3\operatorname{cosec}^3\theta\cot\theta}{2\cos 2\theta}$ (1)

b) find $\theta$ when $y = 8$

$8 = \operatorname{cosec}^3\theta$

$\operatorname{cosec}\theta = 2$

$\sin\theta = \frac{1}{2}$ (1)




$0 < \theta < \frac{\pi}{2}$

$\Rightarrow \theta = \frac{\pi}{6}$

$\theta = \frac{\pi}{6} \Rightarrow \frac{dy}{dx} = \frac{-3\operatorname{cosec}^3\frac{\pi}{6}\cot\frac{\pi}{6}}{2\cos\frac{2\pi}{6}}$ (1)

$\frac{dy}{dx} = -24\sqrt{3}$ (1)

**2.**

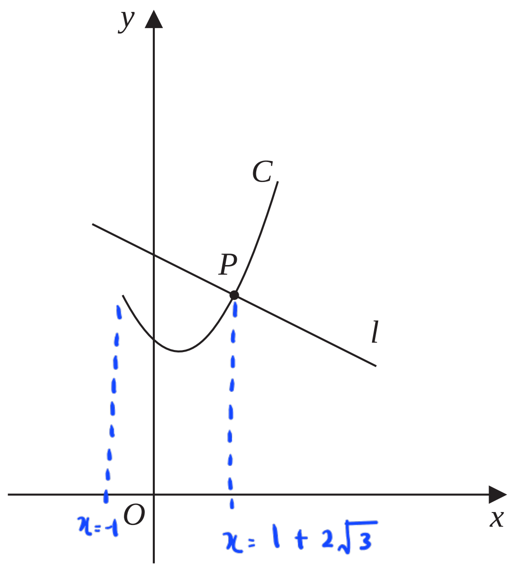



**Figure 6**

Figure 6 shows a sketch of the curve $C$ with parametric equations

$$x = 2\tan t + 1 \qquad y = 2\sec^2 t + 3 \qquad -\frac{\pi}{4} \leqslant t \leqslant \frac{\pi}{3}$$

The line $l$ is the normal to $C$ at the point $P$ where $t = \frac{\pi}{4}$

(a) Using parametric differentiation, show that an equation for $l$ is

$$y = -\frac{1}{2}x + \frac{17}{2}$$

**(5)**

(b) Show that all points on $C$ satisfy the equation

$$y = \frac{1}{2}(x-1)^2 + 5$$

**(2)**

The straight line with equation

$$y = -\frac{1}{2}x + k \qquad \text{where } k \text{ is a constant}$$

intersects $C$ at two distinct points.

(c) Find the range of possible values for $k$.

**(5)**

a) $x = 2(\tan t) + 1$

$\frac{dx}{dt} = 2\sec^2 t$ (1)

$y = 2(\sec t)^2 + 3$

$\frac{dy}{dt} = 4(\sec t)\sec t \tan t = 4\sec^2 t \tan t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\sec^2 t \tan t}{2\sec^2 t} = 2\tan t$ (1)

when $t = \frac{\pi}{4}$, $x = 2\tan\frac{\pi}{4} + 1 = 3$

$y = 2\sec^2\frac{\pi}{4} + 3 = 7$ (1)

$\frac{dy}{dx} = 2\tan\frac{\pi}{4} = 2$ (gradient of tangent)

$\therefore$ gradient of normal is $-\frac{1}{2}$.

$\therefore$ equation of normal : $y - 7 = -\frac{1}{2}(x - 3)$ (1)

$l: \ y = -\frac{1}{2}x + \frac{17}{2}$ (1)

b) from point $x$ : $x = 2\tan t + 1$

$\tan t = \frac{x-1}{2}$

$\tan^2 t = \left(\frac{x-1}{2}\right)^2$ — (1)

from point $y$ : $y = 2\sec^2 t + 3$ (1)

$= 2(1 + \tan^2 t) + 3$

$y = 2\tan^2 t + 5$ — (2)

substitute ① into ②

$y = 2\left(\frac{x-1}{2}\right)^2 + 5$

$y = \frac{1}{2}(x-1)^2 + 5$ — ③ (shown) ①

c) domain of ③ : $2\tan\left(-\frac{\pi}{4}\right) + 1 \leq x \leq 2\tan\left(\frac{\pi}{3}\right) + 1$

$-1 \leq x \leq 1 + 2\sqrt{3}$

$y = -\frac{1}{2}x + k$ — ④

simultaneous equation of ③ and ④

$\frac{1}{2}(x-1)^2 + 5 = -\frac{1}{2}x + k$

$(x-1)^2 + 10 = -x + 2k$

$x^2 - 2x + 1 + 10 = -x + 2k$

$x^2 - x + (11 - 2k) = 0$

$b^2 - 4ac > 0$ : $(-1)^2 - 4(1)(11-2k) > 0$ ①

$8k - 43 > 0$

$k > \frac{43}{8}$ ①

when $x = -1$ : $(-1)^2 - (-1) + (11 - 2k) = 0$ (1)

$2k = 13$

$k = \frac{13}{2}$ (1)

$\therefore$ range of values of $k$ is : $\frac{43}{8} < k \leq \frac{13}{2}$ (1)