1.

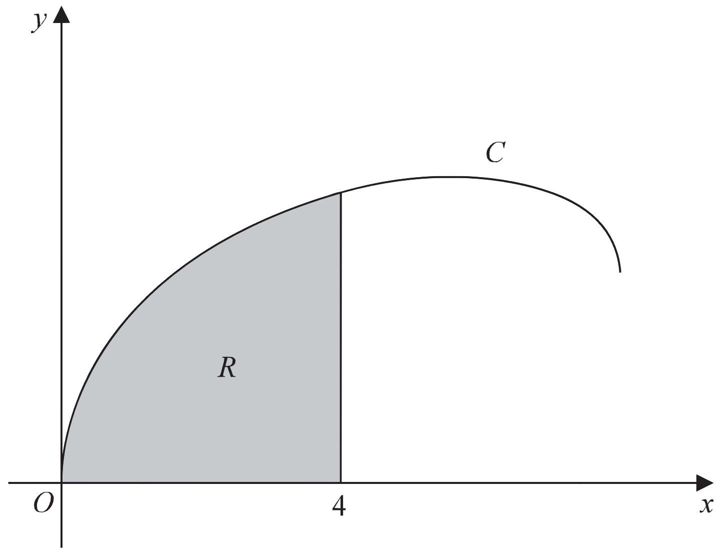


Figure 6

Figure 6 shows a sketch of the curve C with parametric equations

$$x = 8\sin^2 t \qquad y = 2\sin 2t + 3\sin t \qquad 0 \leqslant t \leqslant \frac{\pi}{2}$$

The region R, shown shaded in Figure 6, is bounded by C, the x-axis and the line with equation $x = 4$

(a) Show that the area of R is given by

$$\int_0^a \left(8 - 8\cos 4t + 48\sin^2 t\cos t\right)\mathrm{d}t$$

where a is a constant to be found.

(5)

(b) Hence, using algebraic integration, find the exact area of R.

(4)

(a) $R = \int_q^p y \frac{dx}{dt}\, dt$

$x = 8\sin^2 t$

$\frac{dx}{dt} = 8 \times 2\sin t\cos t$

$= 16\sin t\cos t$

$\frac{d}{dx}\sin^2 x = 2\sin x\cos x$ using the chain rule with $u = \sin x$

$y \times \frac{dx}{dt} = (2\sin 2t + 3\sin t) \times 16\sin t\cos t$ (1)

$$y \times \frac{dx}{dt} = \left[2(\sin t\cos t + \cos t\sin t) + 3\sin t\right] \times 16\sin t\cos t$$

$$= (4\sin t\cos t + 3\sin t) \times 16\sin t\cos t$$

$$= (64\sin^2 t\cos t^2 + 48\sin^2 t\cos t)$$

$$R = \int_0^a y\frac{dx}{dt}\,dt = \int_0^a 64\sin^2 t\cos t^2 + 48\sin^2 t\cos t\;dt \quad (1)$$

We need to find a way to simplify this into the required form.

$$\cos 4t = 2\cos^2 2t - 1$$
$$= 2(1-\sin^2 2t) - 1$$
$$= 2 - 2\sin^2 2t - 1$$

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

$$= 1 - 2\sin^2 2t$$
$$= 1 - 2(\sin 2t \sin 2t)$$
$$= 1 - 2(2\sin t\cos t \times 2\sin t\cos t)$$

using $\sin(2x) = 2\sin x\cos x$

$$= 1 - 2(4\sin^2 t\cos^2 t)$$
$$= 1 - 8\sin^2 t\cos^2 t \quad (1)$$

$$8\sin^2 t\cos^2 t = 1 - \cos 4t$$

$64\sin^2 t\cos^2 t = 8(1-\cos 4t)$

$$R = \int_0^a 8 - 8\cos 4t + 48\sin^2 t\cos t\;dt \quad (1)$$

$$a = \frac{\pi}{4} \quad (1)$$

Finding the new domain :

$$R = \int_0^4 y\,dx = \int y\,\frac{dx}{dt}\,.\,dt$$

$$x = 8\sin^2 t \longrightarrow \frac{dx}{dt} = 16\sin t\cos t$$

when $x = 0$, $8\sin^2 t = 0$, Hence, $t = 0$

when $x = 4$, $8\sin^2 t = 4 \longrightarrow \sin^2 t = \frac{1}{2}$

$$t = \sin^{-1}\sqrt{\tfrac{1}{2}} = \pi/4$$

(b) $\int_0^{\frac{\pi}{4}} 8 - 8\cos 4t + 48\sin^2 t\cos t \ dt = 8t - 2\sin 4t + 16\sin^3 t$ (2)

$$\left[8t - 2\sin 4t + 16\sin^3 t\right]_0^{\frac{\pi}{4}} = \left[8\left(\frac{\pi}{4}\right) - 2\sin\left(4\times\frac{\pi}{4}\right) + 16\sin^3\left(\frac{\pi}{4}\right)\right]$$

$$- \left[8(0) - 2\sin(4\times 0) + 16\sin^3(0)\right]$$ (1)

$$= 2\pi + 4\sqrt{2}$$ (1)

2. A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{7xe^{x}}{\sqrt{e^{3x} - 2}} \qquad x > \ln \sqrt[3]{2}$$

(a) Show that

$$\mathrm{f}'(x) = \frac{7e^{x}(e^{3x}(2 - x) + Ax + B)}{2(e^{3x} - 2)^{\frac{3}{2}}}$$

where A and B are constants to be found.

(5)

(b) Hence show that the x coordinates of the turning points of the curve are solutions of the equation

$$x = \frac{2e^{3x} - 4}{e^{3x} + 4}$$

(2)

The equation $x = \dfrac{2e^{3x} - 4}{e^{3x} + 4}$ has two positive roots α and β where $\beta > \alpha$

A student uses the iteration formula

$$x_{n+1} = \frac{2e^{3x_n} - 4}{e^{3x_n} + 4}$$

in an attempt to find approximations for α and β

Diagram 1 shows a plot of part of the curve with equation $y = \dfrac{2e^{3x} - 4}{e^{3x} + 4}$ and part of the line with equation $y = x$

Using Diagram 1 on page 42

(c) draw a staircase diagram to show that the iteration formula starting with $x_1 = 1$ can be used to find an approximation for β

(1)

Use the iteration formula with $x_1 = 1$, to find, to 3 decimal places,

(d) (i) the value of x_2

(ii) the value of β

(3)

Using a suitable interval and a suitable function that should be stated

(e) show that $\alpha = 0.432$ to 3 decimal places.

(2)

a) $f(x) = \dfrac{7xe^{x}}{(e^{3x}-2)^{1/2}}$

$\dfrac{d}{dx}(7xe^{x})$: let $u = 7x$ $\quad v = e^{x}$

$\dfrac{du}{dx} = 7$ 1 $\quad \dfrac{dv}{dx} = e^{x}$ 1

$\dfrac{du}{dx}v + \dfrac{dv}{dx}u = 7e^{x} + 7xe^{x}$

$\dfrac{d}{dx}\left((e^{3x}-2)^{1/2}\right) = \dfrac{1}{2} \times 3e^{3x} \times (e^{3x}-2)^{-1/2}$ 1

$= \dfrac{3}{2}e^{3x}(e^{3x}-2)^{-1/2}$

$f(x) = \dfrac{7xe^{x}}{(e^{3x}-2)^{1/2}}$

using quotient rule

1

$f'(x) = \dfrac{(e^{3x}-2)^{1/2}(7e^{x}+7xe^{x}) - 7xe^{x}\left(\frac{3}{2}e^{3x}(e^{3x}-2)^{-1/2}\right)}{e^{3x}-2}$

$= \dfrac{7(e^{3x}-2)^{-1/2}\left[e^{x}(e^{3x}-2)(1+x) - \frac{3}{2}xe^{x}e^{3x}\right]}{e^{3x}-2}$

factoring out $7(e^{3x}-2)^{-1/2}$

$= \dfrac{7e^{x}\left[(e^{3x}-2)(1+x) - \frac{3}{2}xe^{3x}\right]}{(e^{3x}-2)^{3/2}}$

moving $(e^{3x}-2)^{-1/2}$ to the denominator

factoring out e^{x}

Only use the copy of Diagram 1 if you need to redraw your answer to part (c).

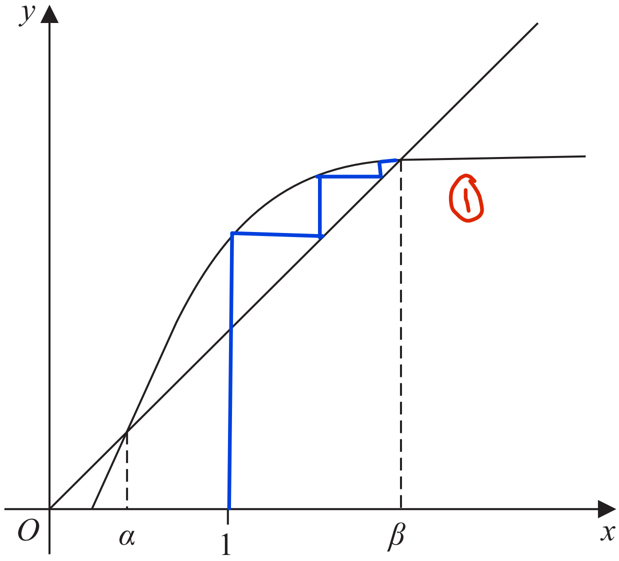
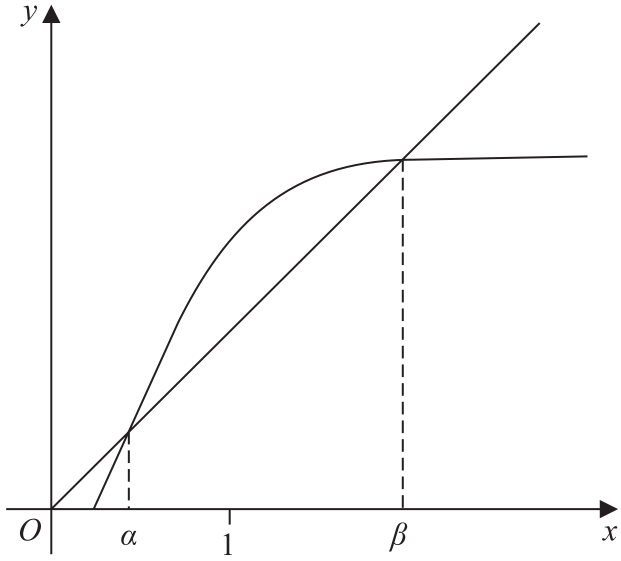


Diagram 1



copy of Diagram 1

expanding bracket

$$f'(x) = \frac{7e^{x}\left[e^{3x} + xe^{3x} - 2 - 2x - \frac{3}{2}xe^{3x}\right]}{(e^{3x}-2)^{3/2}}$$

collecting like terms

$$= \frac{7e^{x}\left[e^{3x} - \frac{1}{2}xe^{3x} - 2x - 2\right]}{(e^{3x}-2)^{3/2}}$$

multiplying top and bottom by 2

$$= \frac{7e^{x}(2e^{3x} - xe^{3x} - 4x - 4)}{2(e^{3x}-2)^{3/2}}$$

(1)

$$= \frac{7e^{x}(e^{3x}(2-x) - 4x - 4)}{2(e^{3x}-2)^{3/2}}$$ as required.

b) turning points have $f'(x)=0$

$$\Rightarrow \frac{7e^{x}\left(e^{3x}(2-x)-4x-4\right)}{2(e^{3x}-2)^{3/2}}=0$$

$e^{x}\neq 0$, multiply by $2(e^{3x}-2)^{3/2}$

$$e^{3x}(2-x)-4x-4=0$$

$$2e^{3x}-xe^{3x}-4x-4=0$$

$$x(e^{3x}+4)=2e^{3x}-4 \quad (1)$$

$$x=\frac{2e^{3x}-4}{e^{3x}+4} \quad (1)$$

c) drawn on diagram

d) $$x_{n+1}=\frac{2e^{3x_n}-4}{e^{3x_n}+4}$$

(i) $$x_2=\frac{2e^{3}-4}{e^{3}+4}=1.50177\ldots \quad (1)$$

$$=1.502 \text{ (3dp)} \quad (1)$$

(ii) $\beta = 1.96757\ldots$

$= 1.968$ (3dp) (1)

e) α is a solution of $x=\dfrac{2e^{3x}-4}{e^{3x}+4}$

$\therefore$ a solution of $\dfrac{2e^{3x}-4}{e^{3x}+4}-x=0$

so define $h(x) = \dfrac{2e^{3x}-4}{e^{3x}+4} - x$, so $h(\alpha)=0$.

$h(0.4315) = -0.000297\ldots < 0$

$h(0.4325) = 0.000947 > 0$ 1

- since there is a change of sign
- and $h(x)$ is continuous
- $\alpha = 0.432$ (to 3dp) 1

3. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

A curve has equation

$$x^3 + 2xy + 3y^2 = 47$$

(a) Find $\frac{dy}{dx}$ in terms of x and y

(4)

The point $P(-2, 5)$ lies on the curve.

(b) Find the equation of the normal to the curve at P, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers to be found.

(3)

a) $x^3 + 2xy + 3y^2 = 47$

$3x^2 + 2y + 2x\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$ (1) (1)

$\frac{dy}{dx}(2x + 6y) = -3x^2 - 2y$ (1)

$\frac{dy}{dx} = -\frac{3x^2 + 2y}{2x + 6y}$ (1)

b) $P(-2, 5)$

$\left.\frac{dy}{dx}\right|_{\substack{x=-2 \\ y=5}} = -\frac{3(-2)^2 + 2(5)}{2(-2) + 6(5)} = -\frac{11}{13}$ (1)

$\therefore \; m_{normal} = \frac{13}{11}$

$y - 5 = \frac{13}{11}(x + 2)$ (1)

$11y - 55 = 13x + 26$

$13x - 11y + 81 = 0$ (1)