Questions

Q1.

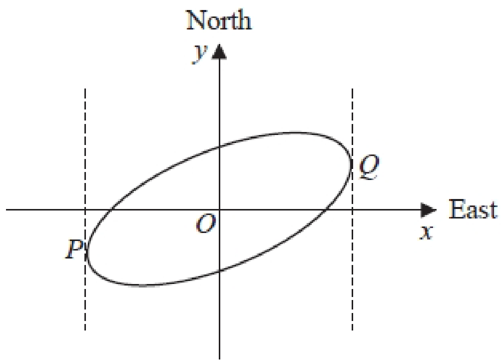

Figure 4

Figure 4 shows a sketch of the curve with equation $x^2 - 2xy + 3y^2 = 50$

(a) Show that $\frac{dy}{dx} = \frac{y - x}{3y - x}$

(4)

The curve is used to model the shape of a cycle track with both *x* and *y* measured in km.

The points *P* and *Q* represent points that are furthest west and furthest east of the origin *O*, as shown in Figure 4.

Using part (a),

(b) find the exact coordinates of the point *P*.

(5)

(c) Explain briefly how to find the coordinates of the point that is furthest north of the origin *O*. (You **do not** need to carry out this calculation).

(1)

(Total for question = 10 marks)

Q2.

The curve *C* has equation

$$x^2 \tan y = 9 \qquad 0 < y < \frac{\pi}{2}$$

(a) Show that

$$\frac{dy}{dx} = \frac{-18x}{x^4 + 81}$$

(4)

(b) Prove that *C* has a point of inflection at $x = \sqrt[4]{27}$

(3)

(Total for question = 7 marks)

Q3.

The curve C has equation

$$px^3 + qxy + 3y^2 = 26$$

where p and q are constants.

(a) Show that

$$\frac{dy}{dx} = \frac{apx^2 + bqy}{qx + cy}$$

where a, b and c are integers to be found.

(4)

Given that

- the point P (–1, – 4) lies on C
- the normal to C at P has equation $19x + 26y + 123 = 0$

(b) find the value of p and the value of q.

(5)

(Total for question = 9 marks)

Q4.

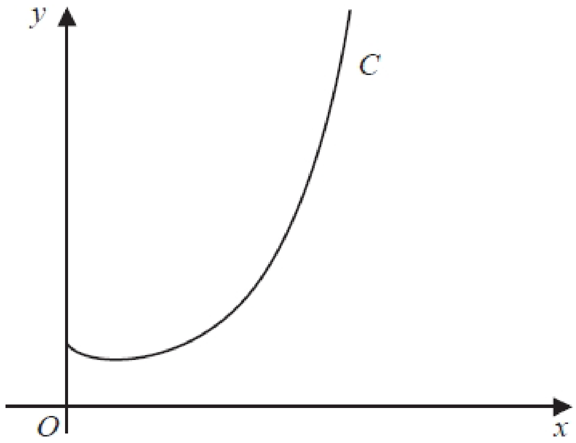

Figure 8

Figure 8 shows a sketch of the curve *C* with equation $y = x^x$, $x > 0$

(a) Find, by firstly taking logarithms, the *x* coordinate of the turning point of *C*.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

The point *P*(α, 2) lies on *C*.

(b) Show that $1.5 < \alpha < 1.6$

(2)

A possible iteration formula that could be used in an attempt to find *α* is

$$x_{n+1} = 2x_n^{1-x_n}$$

Using this formula with $x_1 = 1.5$

(c) find x_4 to 3 decimal places,

(2)

(d) describe the long-term behaviour of x_n

(2)

(Total for question = 11 marks)

Mark Scheme

Q1.

Question	Scheme	Marks	AOs
(a)	Either $3y^2 \to Ay\frac{dy}{dx}$ or $2xy \to 2x\frac{dy}{dx}+2y$	M1	2.1
	$2x-2x\frac{dy}{dx}-2y+6y\frac{dy}{dx}=0$	A1	1.1b
	$(6y-2x)\frac{dy}{dx}=2y-2x$	M1	2.1
	$\frac{dy}{dx}=\frac{2y-2x}{6y-2x}=\frac{y-x}{3y-x}$ *	A1*	1.1b
		(4)	
(b)	$\left(\text{At } P \text{ and } Q \frac{dy}{dx}\to\infty\Rightarrow\right)$ Deduces that $3y-x=0$	M1	2.2a
	Solves $y=\frac{1}{3}x$ and $x^2-2xy+3y^2=50$ simultaneously	M1	3.1a
	$\Rightarrow x=(\pm)5\sqrt{3}$ OR $\Rightarrow y=(\pm)\frac{5}{3}\sqrt{3}$	A1	1.1b
	Using $y=\frac{1}{3}x \Rightarrow x=..$ AND $y=..$	dM1	1.1b
	$P=\left(-5\sqrt{3},-\frac{5}{3}\sqrt{3}\right)$	A1	2.2a
		(5)	
(c)	Explains that you need to solve $y=x$ and $x^2-2xy+3y^2=50$ simultaneously and choose the positive solution	B1ft	2.4
		(1)	
			(10 marks)

Notes:

(a)

M1: For selecting the appropriate method of differentiating either $3y^2 \to Ay\frac{dy}{dx}$ or $2xy \to 2x\frac{dy}{dx}+2y$

It may be quite difficult awarding it for the product rule but condone $-2xy \to -2x\frac{dy}{dx}+2y$ unless you see evidence that they have used the incorrect law vu'-uv'

A1: Fully correct derivative $2x-2x\frac{dy}{dx}-2y+6y\frac{dy}{dx}=0$

Allow attempts where candidates write $2x\text{d}x-2x\text{d}y-2y\text{d}x+6y\text{d}y=0$

but watch for students who write $\frac{dy}{dx}=2x-2x\frac{dy}{dx}-2y+6y\frac{dy}{dx}$ This, on its own, is A0 unless you are convinced that this is just their notation. Eg $\frac{dy}{dx}=2x-2x\frac{dy}{dx}-2y+6y\frac{dy}{dx}=0$

M1: For a valid attempt at making $\frac{dy}{dx}$ the subject. with two terms in $\frac{dy}{dx}$ coming from $3y^2$ and $2xy$

Look for $(\ldots\pm\ldots)\frac{dy}{dx}=\ldots\ldots$ It is implied by $\frac{dy}{dx}=\frac{2y-2x}{6y-2x}$

This cannot be scored from attempts such as $\frac{dy}{dx}=2x-2x\frac{dy}{dx}-2y+6y$ which only has one correct term.

A1*: $\frac{dy}{dx}=\frac{y-x}{3y-x}$ with no errors or omissions.

The previous line $\frac{dy}{dx}=\frac{2y-2x}{6y-2x}$ or equivalent must be seen.

(b)

M1: Deduces that $3y-x=0$ oe

M1: Attempts to find either the x or y coordinates of P and Q by solving their $y=\frac{1}{3}x$ with $x^2-2xy+3y^2=50$ simultaneously. Allow for finding a quadratic equation in x or y and solving to find at least one value for x or y.

This may be awarded when candidates make the numerator = 0 ie using $y=x$

A1: $\Rightarrow x=(\pm)5\sqrt{3}$ OR $\Rightarrow y=(\pm)\frac{5}{3}\sqrt{3}$

dM1: Dependent upon the previous M, it is for finding the y coordinate from their x (or vice versa)

This may also be scored following the numerator being set to 0 ie using $y=x$

A1: Deduces that $P=\left(-5\sqrt{3},-\frac{5}{3}\sqrt{3}\right)$ OE. Allow to be $x=\ldots$ $y=\ldots$

(c)

B1ft: Explains that this is where $\frac{dy}{dx}=0$ and so you need to solve $y=x$ and $x^2-2xy+3y^2=50$ simultaneously and choose the positive solution (or larger solution).

Allow a follow through for candidates who mix up parts (b) and (c)

Alternatively candidates could complete the square $(x-y)^2+2y^2=50$ and state that y would reach a maximum value when $x=y$ and choose the positive solution from $2y^2=50$

Q2.

Question	Scheme	Marks	AOs
(a)	$x^2\tan y = 9 \Rightarrow 2x\tan y + x^2\sec^2 y\dfrac{dy}{dx} = 0$	M1 A1	3.1a 1.1b
	Full method to get $\dfrac{dy}{dx}$ in terms of x using $\sec^2 y = 1+\tan^2 y = 1+f(x)$	M1	1.1b
	$\dfrac{dy}{dx} = \dfrac{-2x\times\dfrac{9}{x^2}}{x^2\left(1+\dfrac{81}{x^4}\right)} = \dfrac{-18x}{x^4+81}$ *	A1*	2.1
		(4)	
(b)	$\dfrac{dy}{dx} = \dfrac{-18x}{x^4+81}$ $\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{-18\times(x^4+81)-(-18x)(4x^3)}{(x^4+81)^2} = \dfrac{54(x^4-27)}{(x^4+81)^2}$ o.e.	M1 A1	1.1b 1.1b
	States that when $x<\sqrt[4]{27} \Rightarrow \dfrac{d^2y}{dx^2}<0$ when $x=\sqrt[4]{27} \Rightarrow \dfrac{d^2y}{dx^2}=0$ **AND** when $x>\sqrt[4]{27} \Rightarrow \dfrac{d^2y}{dx^2}>0$ giving a point of inflection when $x=\sqrt[4]{27}$	A1	2.4
		(3)	
		(7 marks)	
Notes:			

(a)

M1: Attempts to differentiate $\tan y$ implicitly. Eg. $\tan y \rightarrow \sec^2 y\dfrac{dy}{dx}$ or $\cot y \rightarrow -\text{cosec}^2 y\dfrac{dy}{dx}$

You may well see an attempt $\tan y = \dfrac{9}{x^2} \Rightarrow \sec^2 y\dfrac{dy}{dx} = ...$

When a candidate writes $x^2\tan y = 9 \Rightarrow x = 3\tan^{-\frac{1}{2}} y$ the mark is scored for $\tan^{-\frac{1}{2}} y \rightarrow ...\tan^{-\frac{3}{2}} y\sec^2 y$

A1: Correct differentiation $2x\tan y + x^2\sec^2 y\dfrac{dy}{dx} = 0$

Allow also $\sec^2 y\dfrac{dy}{dx} = -\dfrac{18}{x^3}$ or $2x = -9\text{cosec}^2 y\dfrac{dy}{dx}$ amongst others

M1: Full method to get $\dfrac{dy}{dx}$ in terms of x using $\sec^2 y = 1+\tan^2 y = 1+f(x)$

A1*: Proceeds correctly to the given answer of $\dfrac{dy}{dx} = \dfrac{-18x}{x^4+81}$

(b)

M1: Attempts to differentiate the given expression using the product or quotient rule.

For example look for a correct attempt at $\frac{vu'-uv'}{v^2}$ with $u=-18x, v=x^4+81, u'=\pm18, v'=...x^3$

If no method is seen or implied award for $\frac{\pm18\times(x^4+81)\pm18x(ax^3)}{(x^4+81)^2}$

Using the product rule award for $\pm18(x^4+81)^{-1}\pm18x(x^4+81)^{-2}\times cx^3$

A1: Correct **simplified** $\frac{d^2y}{dx^2}=\frac{54(x^4-27)}{(x^4+81)^2}$ o.e. such as $\frac{d^2y}{dx^2}=\frac{54x^4-1458}{(x^4+81)^2}$

Alternatively score for showing that when a correct (unsimplified) $\frac{d^2y}{dx^2}=0\Rightarrow x^4=27\Rightarrow x=\sqrt[4]{27}$

Or for substituting $x=\sqrt[4]{27}$ into an unsimplified but correct $\frac{d^2y}{dx^2}$ and showing that it is 0

A1: Correct explanation with a minimal conclusion and correct second derivative.
See scheme.
It can be also be argued from $x^4<27$, $x^4=27$ and $x^4>27$ provided the conclusion states that the point of inflection is at $x=\sqrt[4]{27}$

Alternatively substitutes values of x either side of $\sqrt[4]{27}$ and at $\sqrt[4]{27}$, into $\frac{d^2y}{dx^2}$, finds all three values and makes a minimal conclusion.

A different method involves finding $\frac{d^3y}{dx^3}$ and showing that $\frac{d^3y}{dx^3}\neq0$ and $\frac{d^2y}{dx^2}=0$ when $x=\sqrt[4]{27}$

FYI $\frac{d^3y}{dx^3}=\frac{23328x^3}{(x^4+81)^3}=0.219$ when $x=\sqrt[4]{27}$

Alternative part (a) using arctan

M1: Sets $y=\arctan\frac{9}{x^2}\rightarrow\frac{dy}{dx}=\frac{1}{1+\left(\frac{9}{x^2}\right)^2}\times...$ where ... could be 1

A2: $y=\arctan\frac{9}{x^2}\rightarrow\frac{dy}{dx}=\frac{1}{1+\left(\frac{9}{x^2}\right)^2}\times-\frac{18}{x^3}$

A1*: $\frac{dy}{dx}=\frac{1}{1+\frac{81}{x^4}}\times-\frac{18}{x^3}=\frac{-18x}{x^4+1}$ showing correct intermediate step and no errors.

Q3.

Question	Scheme	Marks	AOs
(a)	$\frac{\mathrm{d}}{\mathrm{d}x}(3y^2) = 6y\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\frac{\mathrm{d}}{\mathrm{d}x}(qxy) = qx\frac{\mathrm{d}y}{\mathrm{d}x} + qy$	M1	2.1
	$3px^2 + qx\frac{\mathrm{d}y}{\mathrm{d}x} + qy + 6y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$	A1	1.1b
	$(qx+6y)\frac{\mathrm{d}y}{\mathrm{d}x} = -3px^2 - qy \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = ...$	dM1	2.1
	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-3px^2 - qy}{qx+6y}$	A1	1.1b
		(4)	
(b)	$p(-1)^3 + q(-1)(-4) + 3(-4)^2 = 26$	M1	1.1b
	$19x + 26y + 123 = 0 \Rightarrow m = -\frac{19}{26}$	B1	2.2a
	$\frac{-3p(-1)^2 - q(-4)}{q(-1)+6(-4)} = \frac{26}{19}$ or $\frac{q(-1)+6(-4)}{3p(-1)^2 + q(-4)} = -\frac{19}{26}$	M1	3.1a
	$p - 4q = 22,\ 57p - 102q = 624 \Rightarrow p = ..., q = ...$	dM1	1.1b
	$p = 2,\ \ q = -5$	A1	1.1b
		(5)	
			(9 marks)

Notes
(a) M1: For selecting the appropriate method of differentiating: Allow this mark for either $3y^2 \to \alpha y\frac{dy}{dx}$ or $qxy \to \alpha x\frac{dy}{dx} + \beta y$ A1: Fully correct differentiation. Ignore any spurious $\frac{dy}{dx} = ...$ dM1: A valid attempt to make $\frac{dy}{dx}$ the subject with 2 terms only in $\frac{dy}{dx}$ coming from qxy and $3y^2$ **Depends on the first method mark.** A1: Fully correct expression (b) M1: Uses $x = -1$ and $y = -4$ in the equation of C to obtain an equation in p and q B1: Deduces the correct gradient of the given normal. This may be implied by e.g. $19x + 26y + 123 = 0 \Rightarrow y = -\frac{19}{26}x + ... \Rightarrow$ Tangent equation is $y = \frac{26}{19}x + ...$ M1: Fully correct strategy to establish an equation connecting p and q using $x = -1$ and $y = -4$ in their $\frac{dy}{dx}$ and the gradient of the normal. E.g. $(a) = -1 \div \text{their} -\frac{19}{26}$ or $-1 \div (a) = \text{their} -\frac{19}{26}$ dM1: Solves simultaneously to obtain values for p and q. **Depends on both previous method marks.** A1: Correct values **Note that in (b), attempts to form the equation of the normal in terms of p and q and then compare coefficients with $19x + 26y + 123 = 0$ score no marks. If there is any doubt use Review.**

Q4.

Part	Working or answer an examiner might expect to see	Mark	Notes
(a)	$y = x^x \Rightarrow \ln y = x \ln x$	M1	This mark is for a method to find the x-coordinate of the turning point of C by taking logarithms
	$\ln y = x \ln x \Rightarrow \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \ln x + 1$	M1	This mark is given for a method using implicit differentiation
		A1	This mark is given for a correct expression for $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}$
	Setting $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\ln x + 1 = 0$	M1	This mark is given for a method for finding the turning point of C by setting $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
	$x = \mathrm{e}^{-1}$	A1	This mark is given for correctly finding a value for the x-coordinate of the turning point of C
(b)	$1.5^{1.5} = 1.837\ldots$, $1.6^{1.6} = 2.121\ldots$	M1	This mark is given for substituting 1.5 and 1.6 into $y = x^x$
	The curve C contains the points (1.5, 1.8) and (1.6, 2.1). At P, $y = 2$ Since C is continuous, $1.5 < \alpha < 1.6$	A1	This mark is given for a valid explanation that C contains the points (1.5, 1.8) and (1.6, 2.1) and is continuous
(c)	$x_1 = 1.5$ $x_2 = 2 \times 1.5^{-0.5} = 1.633$	M1	This mark is given for finding a correct value for x_2
	$x_3 = 2 \times 1.633^{-0.633} = 1.466$ $x_4 = 2 \times 1.466^{-0.466} = 1.673$	A1	This mark is given for finding a correct value for x_4
(d)	For example: x_n oscillates is periodic is non-convergent	B1	This mark is given for a valid statement about the long-term behaviour of x_n
	between 1 and 2	B1	This mark is given for stating that the behaviour is between 1 and 2
			(Total 11 marks)