Questions

Q1.

Given $y = x(2x + 1)^4$, show that

$$\frac{dy}{dx} = (2x + 1)^n(Ax + B)$$

where n, A and B are constants to be found.

(4)

(Total for question = 4 marks)

Q2.

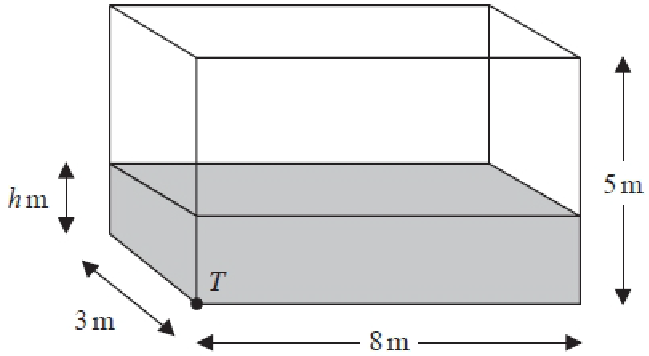

Figure 5

Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.

There is a tap at a point *T* at the bottom of the tank, as shown in Figure 5.

At time *t* minutes after the tap has been opened

- the depth of water in the tank is *h* metres
- water is flowing into the tank at a constant rate of 0.48 m^3 per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h$ m^3 per minute

(a) Show that, according to the model,

$$1200\frac{dh}{dt} = 24 - 5h$$

(4)

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + B\,e^{-kt}$$

where *A*, *B* and *k* are constants to be found.

(6)

Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

(2)

(Total for question = 12 marks)

Q3.

The curve *C* has parametric equations

$$x = 2\cos t, \quad y = \sqrt{3}\cos 2t, \quad 0 \leqslant t \leqslant \pi$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of *t*.

(2)

The point *P* lies on *C* where $t = \frac{2\pi}{3}$

The line *l* is the normal to *C* at *P*.

(b) Show that an equation for *l* is

$$2x - 2\sqrt{3}y - 1 = 0$$

(5)

The line *l* intersects the curve *C* again at the point *Q*.

(c) Find the exact coordinates of *Q*.

You must show clearly how you obtained your answers.

(6)

(Total for question = 13 marks)

Mark Scheme

Q1.

Question	Scheme	Marks	AOs
	Attempts the product and chain rule on $y = x(2x+1)^4$	M1	2.1
	$\frac{dy}{dx} = (2x+1)^4 + 8x(2x+1)^3$	A1	1.1b
	Takes out a common factor $\frac{dy}{dx} = (2x+1)^3\{(2x+1)+8x\}$	M1	1.1b
	$\frac{dy}{dx} = (2x+1)^3(10x+1) \Rightarrow n=3, A=10, B=1$	A1	1.1b
			(4 marks)

Notes:

M1: Applies the product rule to reach $\frac{dy}{dx} = (2x+1)^4 + Bx(2x+1)^3$

A1: $\frac{dy}{dx} = (2x+1)^4 + 8x(2x+1)^3$

M1: Takes out a common factor of $(2x+1)^3$

A1: The form of this answer is given. Look for $\frac{dy}{dx} = (2x+1)^3(10x+1) \Rightarrow n=3, A=10, B=1$

Q2.

Question	Scheme	Marks	AOs
(a)	$\frac{dV}{dt} = 0.48 - 0.1h$	B1	3.1b
	$V = 24h \Rightarrow \frac{dV}{dh} = 24$ or $\frac{dh}{dV} = \frac{1}{24}$	B1	3.1b
	$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV} = \frac{0.48 - 0.1h}{24}$ or e.g. $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} \Rightarrow 0.48 - 0.1h = 24\frac{dh}{dt}$	M1	2.1
	$1200\frac{dh}{dt} = 24 - 5h$ *	A1*	1.1b
		(4)	
(b)	$1200\frac{dh}{dt} = 24 - 5h \Rightarrow \int \frac{1200}{24-5h} dh = \int dt$ $\Rightarrow$ *e.g.* $\alpha \ln(24-5h) = t(+c)$ oe or $1200\frac{dh}{dt} = 24 - 5h \Rightarrow \frac{dt}{dh} = \frac{1200}{24-5h}$ $\Rightarrow$ *e.g.* $t(+c) = \alpha \ln(24-5h)$ oe	M1	3.1a
	$t = -240\ln(24-5h)(+c)$ oe	A1	1.1b
	$t = 0, h = 2 \Rightarrow 0 = -240\ln(24-10) + c \Rightarrow c = ...(240\ln 14)$	M1	3.4
	$t = 240\ln(14) - 240\ln(24-5h)$	A1	1.1b
	$t = 240\ln\frac{14}{24-5h} \Rightarrow \frac{t}{240} = \ln\frac{14}{24-5h} \Rightarrow e^{\frac{t}{240}} = \frac{14}{24-5h}$ $\Rightarrow 14e^{-\frac{t}{240}} = 24 - 5h \Rightarrow h = ...$	ddM1	2.1
	$h = 4.8 - 2.8e^{-\frac{t}{240}}$ oe e.g. $h = \frac{24}{5} - \frac{14}{5}e^{-\frac{t}{240}}$	A1	3.3
		(6)	

(c)	**Examples:** • As $t \to \infty, e^{-\frac{t}{240}} \to 0$ • When $h > 4.8$, $\frac{dV}{dt} < 0$ • Flow in = flow out at max h so $0.1h = 4.8 \rightarrow h = 4.8$ • As $e^{-\frac{t}{240}} > 0$, $h < 4.8$ • $h = 5 \Rightarrow \frac{dV}{dt} = -0.02$ or $\frac{dh}{dt} = -\frac{1}{1200}$ • $\frac{dh}{dt} = 0 \Rightarrow h = 4.8$ • $h = 5 \Rightarrow 4.8 - 2.8e^{-\frac{t}{240}} = 5 \Rightarrow e^{-\frac{t}{240}} < 0$	M1	3.1b
	• The limit for h (according to the model) is 4.8m and the tank is 5m high so the tank will never become full • If $h = 5$ the tank would be emptying so can never be full • The equation can't be solved when $h = 5$	A1	3.2a
		(2)	
			(12 marks)

Notes
(a) B1: Identifies the correct expression for $\frac{dV}{dt}$ according to the model B1: Identifies the correct expression for $\frac{dV}{dh}$ according to the model M1: Applies $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$ or equivalent correct formula with their $\frac{dV}{dt}$ and $\frac{dV}{dh}$ which may be implied by their working A1*: Correct equation obtained with no errors Note that: $\frac{dV}{dt} = 0.48 - 0.1h \Rightarrow \frac{dh}{dt} = \frac{0.48 - 0.1h}{24} \Rightarrow 1200\frac{dh}{dt} = 24 - 5h$ * scores B1B0M0A0. There must be clear evidence where the "24" comes from and evidence of the correct chain rule being applied.

(b)

M1: Adopts a correct strategy by separating the variables correctly or rearranges to obtain $\frac{dt}{dh}$ correctly in terms of h **and** integrates to obtain $t = \alpha \ln(24-5h)(+c)$ or equivalent (condone missing brackets around the "24 – 5h") and + c not required for this mark.

A1: Correct equation in any form and + c not required. Do not condone missing brackets unless they are implied by subsequent work.

M1: Substitutes t = 0 and h = 2 to find their constant of integration (there must have been some attempt to integrate)

A1: Correct equation in any form

ddM1: Uses fully correct log work to obtain h in terms of t.

This depends on <u>both</u> previous method marks.

A1: Correct equation

Note that the marks may be earned in a different order e.g.:

$$t + c = -240\ln(24-5h) \Rightarrow -\frac{t}{240} + d = \ln(24-5h) \Rightarrow Ae^{-\frac{t}{240}} = 24-5h$$

$$t = 0, h = 2 \Rightarrow A = 14 \Rightarrow 14e^{-\frac{t}{240}} = 24-5h \Rightarrow h = 4.8 - 2.8e^{-\frac{t}{240}}$$

Score as M1 A1 as in main scheme then

M1: Correct work leading to $Ae^{\alpha t} = 24-5h$ (must have a constant "A")

A1: $Ae^{-\frac{t}{240}} = 24-5h$

ddM1: Uses t = 0, h = 2 in an expression of the form above to find A

A1: $h = 4.8 - 2.8e^{-\frac{t}{240}}$

(c)

M1: See scheme for some examples

A1: Makes a correct interpretation for their method.

There must be no incorrect working or contradictory statements.

This is not a follow through mark and if their equation in (b) is used it must be correct.

Q3.

Question	Scheme	Marks	AOs
(a)	Attempts $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$	M1	1.1b
	$\frac{dy}{dx} = \frac{\sqrt{3}\sin 2t}{\sin t} \quad (= 2\sqrt{3}\cos t)$	A1	1.1b
		(2)	
(b)	Substitutes $t = \frac{2\pi}{3}$ in $\frac{dy}{dx} = \frac{\sqrt{3}\sin 2t}{\sin t} = (-\sqrt{3})$	M1	2.1
	Uses gradient of normal $= -\frac{1}{dy/dx} = \left(\frac{1}{\sqrt{3}}\right)$	M1	2.1
	Coordinates of $P = \left(-1, -\frac{\sqrt{3}}{2}\right)$	B1	1.1b
	Correct form of normal $y + \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}(x+1)$	M1	2.1
	Completes proof $\Rightarrow 2x - 2\sqrt{3}y - 1 = 0$ *	A1*	1.1b
		(5)	
(c)	Substitutes $x = 2\cos t$ and $y = \sqrt{3}\cos 2t$ into $2x - 2\sqrt{3}y - 1 = 0$	M1	3.1a
	Uses the identity $\cos 2t = 2\cos^2 t - 1$ to produce a quadratic in $\cos t$	M1	3.1a
	$\Rightarrow 12\cos^2 t - 4\cos t - 5 = 0$	A1	1.1b
	Finds $\cos t = \frac{5}{6}$, ~~$-\frac{1}{2}$~~	M1	2.4
	Substitutes their $\cos t = \frac{5}{6}$ into $x = 2\cos t$, $y = \sqrt{3}\cos 2t$,	M1	1.1b
	$Q = \left(\frac{5}{3}, \frac{7}{18}\sqrt{3}\right)$	A1	1.1b
		(6)	
			(13 marks)

Notes:
(a)
M1: Attempts $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$ and achieves a form $k\frac{\sin 2t}{\sin t}$ Alternatively candidates may apply the double angle identity for $\cos 2t$ and achieve a form $k\frac{\sin t\cos t}{\sin t}$
A1: Scored for a correct answer, either $\frac{\sqrt{3}\sin 2t}{\sin t}$ or $2\sqrt{3}\cos t$
(b)
M1: For substituting $t=\frac{2\pi}{3}$ in their $\frac{dy}{dx}$ which must be in terms of t
M1: Uses the gradient of the normal is the negative reciprocal of the value of $\frac{dy}{dx}$. This may be seen in the equation of l.
B1: States or uses (in their tangent or normal) that $P=\left(-1,-\frac{\sqrt{3}}{2}\right)$
M1: Uses their numerical value of $-1/\frac{dy}{dx}$ with their $\left(-1,-\frac{\sqrt{3}}{2}\right)$ to form an equation of the normal at P
A1*: This is a proof and all aspects need to be correct. Correct answer only $2x-2\sqrt{3}y-1=0$

(c)
M1: For substituting $x=2\cos t$ and $y=\sqrt{3}\cos 2t$ into $2x-2\sqrt{3}y-1=0$ to produce an equation in t. Alternatively candidates could use $\cos 2t=2\cos^2 t-1$ to set up an equation of the form $y=Ax^2+B$.
M1: Uses the identity $\cos 2t=2\cos^2 t-1$ to produce a quadratic equation in $\cos t$ In the alternative method it is for combining their $y=Ax^2+B$ with $2x-2\sqrt{3}y-1=0$ to get an equation in just one variable
A1: For the correct quadratic equation $12\cos^2 t-4\cos t-5=0$ Alternatively the equations in x and y are $3x^2-2x-5=0$ $12\sqrt{3}y^2+4y-7\sqrt{3}=0$
M1: Solves the quadratic equation in $\cos t$ (or x or y) and rejects the value corresponding to P.
M1: Substitutes their $\cos t=\frac{5}{6}$ or their $t=\arccos\left(\frac{5}{6}\right)$ in $x=2\cos t$ and $y=\sqrt{3}\cos 2t$ If a value of x or y has been found it is for finding the other coordinate.
A1: $Q=\left(\frac{5}{3},\frac{7}{18}\sqrt{3}\right)$. Allow $x=\frac{5}{3}, y=\frac{7}{18}\sqrt{3}$ but do not allow decimal equivalents.