Differentiation Cheat Sheet

Previously in Pure Year 1, you only learnt how to differentiate simple expressions, such as $2x^2$. In this chapter, we will learn more rules and methods that will enable us to differentiate much more complicated functions.

Differentiating **sinx** and **cosx**

You need to be know the following two results, and be able to prove $\frac{d}{dx}[sinx] = cosx$ and $\frac{d}{dx}[cosx] = -sinx$ from first principles.

- $\frac{d}{dx}[\sin kx] = kcos\ kx$
- $\frac{d}{dx}[\cos kx] = -ksin\ kx$

<u>Example 1</u>: Prove, from first principles, that the derivative of $sinx$ is $cosx$. You may assume that as $h \to 0$, $\frac{sinh}{h} \to 1$ and $\frac{cosh-1}{h} \to 0$.

Whenever you want to prove a result from first principles, we must use the following definition of the derivative:

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

Letting $f(x) = sinx$ and using the above definition:

Using the addition formula, $sin(A+B)$

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] = \lim_{h\to 0}\left[\frac{sin(x+h)-sinx}{h}\right] = \lim_{h\to 0}\left[\frac{sinxcosh + cosxsinh - sinx}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{sinx(cosh-1)+cosxsinh}{h}\right] = \lim_{h\to 0}\left[\frac{sinx(cosh-1)}{h} + \frac{cosx(\sinh)}{h}\right]$$

$$= \lim_{h\to 0}[sinx(0) + cosx(1)] = \lim_{h\to 0}[cosx] = cosx$$

Factorising out $sinx$ in the numerator, and then splitting up the fraction

Using the assumptions given to us in the question

The proof for the derivative of $cosx$ is very similar. We would start by instead letting $f(x) = cosx$.

Differentiating exponentials and logarithms

You should also remember the following results:

- $\frac{d}{dx}[e^{kx}] = ke^{kx}$
- $\frac{d}{dx}[a^{kx}] = a^{kx}(k\ln a)$
- $\frac{d}{dx}[\ln x] = \frac{1}{x}$

You need to know how to prove the second result.

Example 2: Show that the derivative of a^x is $a^x(\ln a)$.

We start by letting $y = a^x$. Now using the following property of exponentials: $e^{lnf(x)} = f(x)$,

$\Rightarrow y = e^{\ln(a^x)} = e^{x(lna)}$

$\therefore \frac{dy}{dx} = (\ln a) \times e^{x(lna)}$ — Using the result: $\frac{d}{dx}[e^{kx}] = ke^{kx}$ with $k = \ln a$

But since we said in the first line that $y = e^{x(lna)}$, we have that

$\frac{dy}{dx} = (\ln a) \times y = \frac{dy}{dx} = (\ln a) \times a^x$ as required.

The chain rule

The chain rule is a powerful method used to differentiate composite functions (i.e. expressions where one function is contained in another function). An example of such a function would be e^{x^2-3x+1}, where the function x^2-3x+1 is contained inside the function e^x. This rule allows us to differentiate seemingly complex expressions with ease and plays a pivotal role in this chapter.

The chain rule is:

- $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, where y is a function of u and u is another function of x.

We can write this in function notation, which tends to be easier to use in application:

- If $y = [f(x)]^n$, then $\frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$ [1]
- If $y = f[g(x)]$ then $\frac{dy}{dx} = f'[g(x)] \times g'(x)$ [2]

When differentiating functions that are not of the form $y = f(x)$, the following case of the chain rule is useful:

- $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

Here are two examples showing the chain rule in action:

<u>Example 3:</u> Differentiate $(sinx + cosx)^5$

This is of the form [1], where $f(x) = sinx + cosx$ and $n = 5$.

$\Rightarrow \frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$

$= 5[sinx + cosx]^4 \times (cosx - sinx)$

$= 5[sinx + cosx]^4(cosx - sinx)$

<u>Example 4:</u> Differentiate e^{x^2-4x+2}

This is of the form [2], where $g(x) = x^2 - 4x + 2$ and $f(x) = e^x$.

$\Rightarrow f'(x) = e^x$ and $g'(x) = 2x - 4$

$\therefore \frac{dy}{dx} = f'[g(x)] \times g'(x)$

$= f'[x^2 - 4x + 2] \times (2x - 4)$

$= e^{x^2-4x+2} \times (2x-4) = (2x-4)e^{x^2-4x+2}$

The product rule

When we want to differentiate an expression that is a product of two functions, we can use the product rule. The product rule is:

- If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ $\xrightarrow{\text{A more convenient way to write this is}}$ $\frac{dy}{dx} = uv' + vu'$

where u and v are both functions of x

where u' and v' are derivatives of u and v with respect to x.

The quotient rule

When we want to differentiate an expression that is the quotient of two functions, we can use the quotient rule. The quotient rule is:

- If $y = uv$, then $\frac{dy}{dx} = \frac{v\frac{dy}{dx} - u\frac{dy}{dx}}{v^2}$ $\xrightarrow{\text{A more convenient way to write this is}}$ $\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$

where u and v are both functions of x

where u' and v' are derivatives of u and v with respect to x.

Note that the product rule could always be used in place of the quotient rule. For example, to differentiate $\frac{x}{(2x+1)^2}$, we could use the quotient rule with $u = x,\ v = (2x+1)^2$, but if we rewrite the expression as $x(2x+1)^{-2}$ then we can also use the product rule with $u = x,\ v = (2x+1)^{-2}$.

Differentiating trigonometric functions

You need to learn and be able to prove the following results:

- $\frac{d}{dx}[\tan kx] = ksec^2\ kx$
- $\frac{d}{dx}[\text{cosec}\ kx] = -kcosec\ kx\cot kx$
- $\frac{d}{dx}[\cot kx] = -kcosec^2\ kx$
- $\frac{d}{dx}[\sec kx] = ksec\ kx\tan kx$

To prove any of the above, you can rewrite the functions in terms of $sinx$ and $cosx$ and differentiate using the chain rule and product/quotient rules.

<u>Example 5:</u> Prove that the derivative of $kcosecx$ is $-kcosec\ kx\cot kx$.

$coseckx = \frac{1}{\sin kx} = (\sin kx)^{-1}$

By the chain rule [1], we have that $\frac{dy}{dx} = -1(\sin kx)^{-2} \times (kcos\ kx) = -\frac{kcos\ kx}{sin^2 kx}$

$= -k\frac{cos\ kx}{\sin kx} \times \frac{1}{\sin kx} = -kcot\ kx\ cosec\ kx$ as required.

The method for proving the other results is very similar.

Parametric differentiation

To find the gradient of a function given in parametric form, you can use the following case of the chain rule:

- $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Example 6: Given that $x = 2sint, y = 4t$, find $\frac{dy}{dx}$ at $t = \pi$.

$\frac{dx}{dt} = 2cost$ and $\frac{dy}{dt} = 4 \Rightarrow \frac{dy}{dx} = \frac{4}{2cost}$

at $t = \pi$, $\frac{dy}{dx} = \frac{4}{2\cos\pi} = \frac{4}{-2} = -2$

Implicit differentiation

Equations of the form $y = f(x)$ are known as explicit equations. We have seen how to differentiate functions of this form many times. However, not all equations are in the form $y = f(x)$ and some cannot even be rearranged into this form. These are known as implicit equations. To differentiate equations of this type, we can use implicit differentiation.

Implicit differentiation is just another case of the chain rule, and can be summarised as:

- $\frac{d}{dx}[f(y)] = f'(y)\frac{dy}{dx}$

This means that if we are differentiating a function of y with respect to x, we simply differentiate the function with respect to y and multiply the result by $\frac{dy}{dx}$.

<u>Example 7:</u> Find $\frac{dy}{dx}$, given that $2x^2 - 3^y - 4xy = 0$.

We differentiate both sides with respect to x: $\frac{d}{dx}[2x^2 - 3^y - 4xy] = \frac{d}{dx}[0]$

$\frac{d}{dx}[2x^2] = 4x$

$\frac{d}{dx}[3^y] = 3^y\ln 3 \times \frac{dy}{dx}$

$\frac{d}{dx}[4xy] = 4y + 4x\frac{dy}{dx}$ — Using the product rule: $u = 4x\ \ u' = 4$, $v = y\ \ v' = \frac{dy}{dx}$

$4x + 3^y\ln 3 \times \frac{dy}{dx} + 4y + 4x\frac{dy}{dx} = 0$

$\frac{dy}{dx}[3^y\ln 3 + 4x] = -4x - 4y$ factorising out $\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-4x-4y}{3^y\ln 3 + 4x}$ making $\frac{dy}{dx}$ the subject

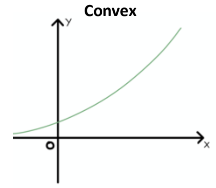
Using second derivatives

You need to be able to use the second derivative to figure out whether a curve is concave or convex on a given interval.

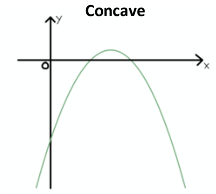
- The function $f(x)$ is concave on a given interval if and only if $f''(x) \leq 0$ for every value of x in that interval.
- The function $f(x)$ is convex on a given interval if and only if $f''(x) \geq 0$ for every value of x in that interval.

Here are two examples showing what concave and convex functions look like.

Convex



Concave



You also need to know the definition of a point of inflection and be able to determine if a given point is a point of inflection.

- A point of inflection is a point where $f''(x)$ changes sign. To determine a point of inflection, you must show that $f''(x) = 0$ at that point and that $f''(x)$ has opposing signs on either side of the point.

Rates of change

An equation involving a derivative is known as a differential equation. You need to be able to form differential equations using information given in a question. You also need to be able to apply the chain rule to problems involving rates of change, where there are more than two variables involved.

Here are some general tips to remember when dealing with rate of change problems:

- The units given in the question are helpful in interpreting the information given. In the below example, we are told "The rate at which fluid flows in m^3min^{-1} is proportional to…". The units m^3 and min^{-1} represent volume and time which indicates this sentence is telling us about $\frac{dV}{dt}$.
- When a quantity is decreasing with time, remember that your expression for the rate of change should be negative.

<u>Example 8:</u> Fluid flows out of a cylindrical tank with constant cross section. At time t minutes, $t > 0$, the volume of fluid remaining in the tank is $V\ m^3$. The rate at which fluid flows in m^3min^{-1} is proportional to the square root of V.
Show that the depth, h metres, of fluid in the tank satisfies the differential equation $\frac{dh}{dt} = -k\sqrt{h}$, where k is a positive constant.

We are told the rate of fluid flow is proportional to the square root of V, so $\frac{dV}{dt} \alpha \sqrt{V}$. We can write this as:
$\frac{dV}{dt} = -c\sqrt{V} = -c\sqrt{\pi r^2 h}$ since the tank is cylindrical and c is some constant. We have a negative sign because the volume in the tank is decreasing with time, as fluid is flowing out of the tank.

We currently have $\frac{dV}{dt}$, but we want $\frac{dh}{dt}$. This is where we need to use the chain rule to figure out how we can get from $\frac{dh}{dt}$ to $\frac{dV}{dt}$.

By the chain rule, $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$. This tells us we need to find $\frac{dh}{dV}$, then multiply by $\frac{dV}{dt}$. Now recall again that the volume of the tank is given by $V = \pi r^2 h$. Differentiating this equation with respect to h:

$\Rightarrow \frac{dh}{dV} = \pi r^2$, so $\frac{dh}{dV} = \frac{1}{\pi r^2}$.

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV} = c\sqrt{\pi r^2 h} \times \frac{1}{\pi r^2} = \frac{c\sqrt{h}}{\sqrt{\pi r^2}}.$$

But since π, r and c are all constants, we can let $\frac{c}{\sqrt{\pi r^2}} = k$. $\therefore \frac{dh}{dt} = k\sqrt{h}$ for some constant k, as required.