Questions

Q1.

Prove, from first principles, that the derivative of $3x^2$ is $6x$.

(4)

(Total for question = 4 marks)

Q2.

Prove, from first principles, that the derivative of x^3 is $3x^2$

(4)

(Total for question = 4 marks)

Q3.

Given that θ is measured in radians, prove, from first principles, that the derivative of $\sin\theta$ is $\cos\theta$

You may assume the formula for $\sin(A \pm B)$ and that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

(5)

(Total for question = 5 marks)

Q4.

Given that θ is measured in radians, prove, from first principles, that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta) = -\sin\theta$$

You may assume the formula for cos ($A \pm B$) and that as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

(5)

(Total for question = 5 marks)

Q5.

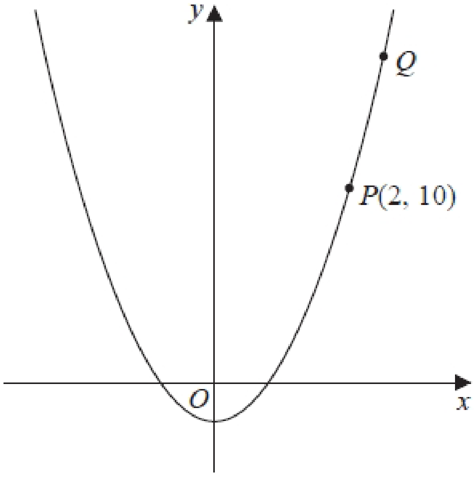

Figure 1

Figure 1 shows part of the curve with equation $y = 3x^2 - 2$

The point $P(2, 10)$ lies on the curve.

(a) Find the gradient of the tangent to the curve at P.

(2)

The point Q with x coordinate $2 + h$ also lies on the curve.

(b) Find the gradient of the line PQ, giving your answer in terms of h in simplest form.

(3)

(c) Explain briefly the relationship between part (b) and the answer to part (a).

(1)

(Total for question = 6 marks)

Mark Scheme

Q1.

Question	Scheme	Marks	AOs
	Considers $\dfrac{3(x+h)^2-3x^2}{h}$	B1	2.1
	Expands $3(x+h)^2=3x^2+6xh+3h^2$	M1	1.1b
	so gradient $=\dfrac{6xh+3h^2}{h}=6x+3h$ or $\dfrac{6x\delta x+3(\delta x)^2}{\delta x}=6x+3\delta x$	A1	1.1b
	States as $h\to 0$, gradient $\to 6x$ so in the limit derivative $=6x$ *	A1*	2.5
			(4 marks)

Notes

B1: gives correct fraction as in the scheme above or $\dfrac{3(x+\delta x)^2-3x^2}{\delta x}$

M1: Expands the bracket as above or $3(x+\delta x)^2=3x^2+6x\delta x+3(\delta x)^2$

A1: Substitutes correctly into earlier fraction and simplifies

A1*: Completes the proof, as above (may use $\delta x \to 0$), considers the limit and states a conclusion with no errors

Q2.

Question	Scheme	Marks	AOs
	Considers $\dfrac{(x+h)^3-x^3}{h}$	B1	2.1
	Expands $(x+h)^3 = x^3+3x^2h+3xh^2+h^3$	M1	1.1b
	so gradient (of chord) $=\dfrac{3x^2h+3xh^2+h^3}{h}=3x^2+3xh+h^2$	A1	1.1b
	States as $h\to 0$, $3x^2+3xh+h^2\to 3x^2$ so derivative $=3x^2$ *	A1*	2.5
		(4 marks)	

B1: Gives the correct fraction for the gradient of the chord either $\dfrac{(x+h)^3-x^3}{h}$ or $\dfrac{(x+\delta x)^3-x^3}{\delta x}$

It may also be awarded for $\dfrac{(x+h)^3-x^3}{x+h-x}$ oe. It may be seen in an expanded form

It does not have to be linked to the gradient of the chord

M1: Attempts to expand $(x+h)^3$ or $(x+\delta x)^3$ Look for two correct terms, most likely $x^3+...+h^3$
This is independent of the B1

A1: Achieves gradient (of chord) is $3x^2+3xh+h^2$ or exact un simplified equivalent such as $3x^2+2xh+xh+h^2$. Again, there is no requirement to state that this expression is the gradient of the chord

A1*: CSO. Requires correct algebra and making a link between the gradient of the chord and the gradient of the curve. See below how the link can be made. The words "gradient of the chord" do not need to be mentioned but derivative, $f'(x)$, $\dfrac{dy}{dx}$, y' should be. Condone invisible brackets for the expansion of $(x+h)^3$ as long as it is only seen at the side as intermediate working.

Requires either

- $f'(x) \underset{\lim h\to 0}{=} \dfrac{(x+h)^3-x^3}{h}=3x^2+3xh+h^2=3x^2$
- Gradient of chord $=3x^2+3xh+h^2$ As $h\to 0$ Gradient of chord tends to the gradient of curve so derivative is $3x^2$
- $f'(x) \underset{\lim h\to 0}{=} 3x^2+3xh+h^2=3x^2$
- Gradient of chord $=3x^2+3xh+h^2$ when $h\to 0$ gradient of curve $=3x^2$
- Do not allow $h=0$ alone without limit being considered somewhere:
  so don't accept $h=0 \Rightarrow f'(x)=3x^2+3x\times 0+0^2=3x^2$

Alternative: B1: Considers $\dfrac{(x+h)^3-(x-h)^3}{2h}$ M1: As above A1: $\dfrac{6x^2h^2+2h^3}{2h}=3x^2+h^2$

Q3.

Question	Scheme	Marks	AOs
	Use of $\dfrac{\sin(\theta+h)-\sin\theta}{(\theta+h)-\theta}$	B1	2.1
	Uses the compound angle identity for $\sin(A+B)$ with $A=\theta,\ B=h$ $\Rightarrow \sin(\theta+h)=\sin\theta\cos h+\cos\theta\sin h$	M1	1.1b
	Achieves $\dfrac{\sin(\theta+h)-\sin\theta}{h}=\dfrac{\sin\theta\cos h+\cos\theta\sin h-\sin\theta}{h}$	A1	1.1b
	$=\dfrac{\sin h}{h}\cos\theta+\left(\dfrac{\cos h-1}{h}\right)\sin\theta$	M1	2.1
	Uses $h\to 0,\ \dfrac{\sin h}{h}\to 1$ and $\dfrac{\cos h-1}{h}\to 0$ Hence the $\lim_{h\to 0}\dfrac{\sin(\theta+h)-\sin\theta}{(\theta+h)-\theta}=\cos\theta$ and the gradient of the chord $\to$ gradient of the curve $\Rightarrow \dfrac{dy}{d\theta}=\cos\theta$ *	A1*	2.5
		(5 marks)	

Notes:

B1: States or implies that the gradient of the chord is $\dfrac{\sin(\theta+h)-\sin\theta}{h}$ or similar such as $\dfrac{\sin(\theta+\delta\theta)-\sin\theta}{\theta+\delta\theta-\theta}$ for a small h or $\delta\theta$

M1: Uses the compound angle identity for $\sin(A+B)$ with $A=\theta,\ B=h$ or $\delta\theta$

A1: Obtains $\dfrac{\sin\theta\cos h+\cos\theta\sin h-\sin\theta}{h}$ or equivalent

M1: Writes their expression in terms of $\dfrac{\sin h}{h}$ and $\dfrac{\cos h-1}{h}$

A1*: Uses correct language to explain that $\dfrac{dy}{d\theta}=\cos\theta$

For this method they should use all of the given statements $h\to 0,\ \dfrac{\sin h}{h}\to 1,$ $\dfrac{\cos h-1}{h}\to 0$ meaning that the $\lim_{h\to 0}\dfrac{\sin(\theta+h)-\sin\theta}{(\theta+h)-\theta}=\cos\theta$

and therefore the gradient of the chord $\to$ gradient of the curve $\Rightarrow \dfrac{dy}{d\theta}=\cos\theta$

Question	Scheme	Marks	AOs
alt	Use of $\dfrac{\sin(\theta+h)-\sin\theta}{(\theta+h)-\theta}$	B1	2.1
	Sets $\dfrac{\sin(\theta+h)-\sin\theta}{(\theta+h)-\theta}=\dfrac{\sin\left(\theta+\frac{h}{2}+\frac{h}{2}\right)-\sin\left(\theta+\frac{h}{2}-\frac{h}{2}\right)}{h}$ and uses the compound angle identity for $\sin(A+B)$ and $\sin(A-B)$ with $A=\theta+\frac{h}{2}$, $B=\frac{h}{2}$	M1	1.1b
	Achieves $\dfrac{\sin(\theta+h)-\sin\theta}{h}=$ $\dfrac{\left[\sin\left(\theta+\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)+\cos\left(\theta+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)\right]-\left[\sin\left(\theta+\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)-\cos\left(\theta+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)\right]}{h}$	A1	1.1b
	$=\dfrac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\times\cos\left(\theta+\frac{h}{2}\right)$	M1	2.1
	Uses $h\to 0,\ \frac{h}{2}\to 0$ hence $\dfrac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\to 1$ and $\cos\left(\theta+\frac{h}{2}\right)\to\cos\theta$ Therefore the $\text{limit}_{h\to 0}\dfrac{\sin(\theta+h)-\sin\theta}{(\theta+h)-\theta}=\cos\theta$ and the gradient of the chord $\to$ gradient of the curve $\Rightarrow \dfrac{dy}{d\theta}=\cos\theta$ *	A1*	2.5
		(5 marks)	

Additional notes:

A1*: Uses correct language to explain that $\dfrac{dy}{d\theta}=\cos\theta$. For this method they should use the (adapted) given statement $h\to 0,\ \frac{h}{2}\to 0$ hence $\dfrac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\to 1$ with $\cos\left(\theta+\frac{h}{2}\right)\to\cos\theta$ meaning that the $\text{limit}_{h\to 0}\dfrac{\sin(\theta+h)-\sin\theta}{(\theta+h)-\theta}=\cos\theta$ and therefore the gradient of the chord $\to$ gradient of the curve $\Rightarrow \dfrac{dy}{d\theta}=\cos\theta$

Q4.

Question	Scheme	Marks	AOs
	$\frac{d}{d\theta}(\cos\theta) = -\sin\theta$; as $h \to 0$, $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$		
	$\frac{\cos(\theta+h)-\cos\theta}{h}$	B1	2.1
	$= \frac{\cos\theta\cos h - \sin\theta\sin h - \cos\theta}{h}$	M1	1.1b
		A1	1.1b
	$= -\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h - 1}{h}\right)\cos\theta$		
	As $h \to 0$, $-\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h - 1}{h}\right)\cos\theta \to -1\sin\theta + 0\cos\theta$	dM1	2.1
	so $\frac{d}{d\theta}(\cos\theta) = -\sin\theta$ *	A1*	2.5
		(5)	
		(5 marks)	

Notes for Question	
B1:	Gives the correct fraction such as $\frac{\cos(\theta+h)-\cos\theta}{h}$ or $\frac{\cos(\theta+\delta\theta)-\cos\theta}{\delta\theta}$ Allow $\frac{\cos(\theta+h)-\cos\theta}{(\theta+h)-\theta}$ o.e. **Note:** $\cos(\theta+h)$ or $\cos(\theta+\delta\theta)$ may be expanded
M1:	Uses the compound angle formula for $\cos(\theta+h)$ to give $\cos\theta\cos h \pm \sin\theta\sin h$
A1:	Achieves $\frac{\cos\theta\cos h - \sin\theta\sin h - \cos\theta}{h}$ or equivalent
dM1:	**dependent on both the B and M marks being awarded** Complete attempt to apply the given limits to the gradient of their chord
Note:	They must isolate $\frac{\sin h}{h}$ and $\left(\frac{\cos h - 1}{h}\right)$, and replace $\frac{\sin h}{h}$ with 1 and replace $\left(\frac{\cos h - 1}{h}\right)$ with 0
A1*:	cso. Uses correct mathematical language of limiting arguments to prove $\frac{d}{d\theta}(\cos\theta) = -\sin\theta$
Note:	Acceptable responses for the final A mark include: • $\frac{d}{d\theta}(\cos\theta) = \lim_{h\to 0}\left(-\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h - 1}{h}\right)\cos\theta\right) = -1\sin\theta + 0\cos\theta = -\sin\theta$ • Gradient of chord $= -\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h - 1}{h}\right)\cos\theta$. As $h \to 0$, gradient of chord tends to the gradient of the curve, so derivative is $-\sin\theta$ • Gradient of chord $= -\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h - 1}{h}\right)\cos\theta$. As $h \to 0$, gradient of *curve* is $-\sin\theta$
Note:	Give final A0 for the following example which shows ***no limiting arguments***: when $h = 0$, $\frac{d}{d\theta}(\cos\theta) = -\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h - 1}{h}\right)\cos\theta = -1\sin\theta + 0\cos\theta = -\sin\theta$
Note:	Do not allow the final A1 for stating $\frac{\sin h}{h} = 1$ or $\left(\frac{\cos h - 1}{h}\right) = 0$ and attempting to apply these
Note:	In this question $\delta\theta$ may be used in place of h
Note:	Condone $f'(\theta)$ where $f(\theta) = \cos\theta$ or $\frac{dy}{d\theta}$ where $y = \cos\theta$ used in place of $\frac{d}{d\theta}(\cos\theta)$

Notes for Question Continued	
Note:	Condone x used in place of θ if this is done consistently
Note:	Give final A0 for • $\frac{d}{d\theta}(\cos x) = \lim_{h\to 0}\left(-\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h-1}{h}\right)\cos\theta\right) = -1\sin\theta + 0\cos\theta = -\sin\theta$ • $\frac{d}{d\theta} = ...$ • Defining $f(x)=\cos\theta$ and applying $f'(x)=...$ • $\frac{d}{dx}(\cos\theta)$
Note:	Give final A1 for a correct limiting argument in x, followed by $\frac{d}{d\theta}(\cos\theta) = -\sin\theta$ e.g. $\frac{d}{d\theta}(\cos x) = \lim_{h\to 0}\left(-\frac{\sin h}{h}\sin x + \left(\frac{\cos h-1}{h}\right)\cos x\right) = -1\sin x + 0\cos x = -\sin x$ $\Rightarrow \frac{d}{d\theta}(\cos\theta) = -\sin\theta$
Note:	Applying $h\to 0,\ \sin h\to h,\ \cos h\to 1$ to give e.g. $\lim_{h\to 0}\left(\frac{\cos\theta\cos h - \sin\theta\sin h - \cos\theta}{h}\right) = \left(\frac{\cos\theta(1) - \sin\theta(h) - \cos\theta}{h}\right) = \frac{-\sin\theta(h)}{h} = -\sin\theta$ is final M0 A0 for incorrect application of limits
Note:	$\lim_{h\to 0}\left(\frac{\cos\theta\cos h - \sin\theta\sin h - \cos\theta}{h}\right) = \lim_{h\to 0}\left(-\frac{\sin h}{h}\sin\theta + \left(\frac{\cos h-1}{h}\right)\cos\theta\right)$ $= \lim_{h\to 0}\left(-(1)\sin\theta + 0\cos\theta\right) = -\sin\theta$. So for not removing $\lim_{h\to 0}$ when the limit was taken is final A0
Note:	**Alternative Method:** Considers $\frac{\cos(\theta+h)-\cos(\theta-h)}{(\theta+h)-(\theta-h)}$ which simplifies to $\frac{-2\sin\theta\sin h}{2h}$

Q5.

Question	Scheme	Marks	AOs
(a)	Attempts to find the value of $\frac{dy}{dx}$ at $x = 2$	M1	1.1b
	$\frac{dy}{dx} = 6x \Rightarrow$ gradient of tangent at P is 12	A1	1.1b
		(2)	
(b)	Gradient $PQ = \frac{3(2+h)^2 - 2 - 10}{(2+h)-2}$ oe	B1	1.1b
	$= \frac{3(2+h)^2 - 12}{(2+h)-2} = \frac{12h + 3h^2}{h}$	M1	1.1b
	$= 12 + 3h$	A1	2.1
		(3)	
(c)	Explains that as $h \to 0,\ 12 + 3h \to 12$ and states that the gradient of the chord tends to the gradient of (the tangent to) the curve	B1	2.4
		(1)	
			(6 marks)

Notes

(a)

M1: Attempts to differentiate, allow $3x^2 - 2 \to ...x$ and substitutes $x = 2$ into their answer

A1: cso $\frac{dy}{dx} = 6x \Rightarrow$ gradient of tangent at P is 12

(b)

B1: Correct expression for the gradient of the chord seen or implied.

M1: Attempts $\frac{\delta y}{\delta x}$, condoning slips, and attempts to simplify the numerator. The denominator must be h

A1: cso $12 + 3h$

(c)

B1: Explains that as $h \to 0,\ 12 + 3h \to 12$ and states that the gradient of the chord tends to the gradient of the curve