Differentiation Cheat Sheet

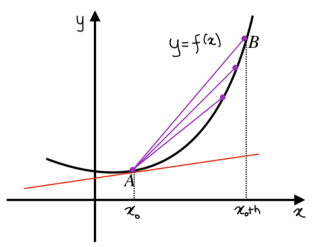
Differentiation is a process that helps us to calculate gradient or slope of a function at different points. It also help us to identify change in one variable with respect to another variable. You will learn real life application of differentiation in this topic

Derivative
Notation:
$\frac{dy}{dx}$ of $f'(x)$ represents derivative of $y = f(x)$ with respect to x

Gradient of curves:
Unlike straight lines, the gradient of a curve changes constantly. The gradient of a curve at any given point is the same as calculating gradient of the tangent at that given point. You can find exact gradient of a curve at any given point using derivatives.

- The gradient of a curve at any given point → • calculating gradient of the tangent at that given point → • Finding derivative at that given point

Finding derivative:
In this section you will learn how to find derivative of a function (i.e. exact gradient of a curve or function at a given point)
Consider the following figure for a curve $y = f(x)$



As point B moves closer to point A, the gradient of chord AB gets closer to the gradient of tangent to the curve at A.
The coordinate of A is $(x_0, f(x_0))$ and B is $(x_0 + h, f(x_0 + h))$. So, the gradient of AB is $\frac{f(x_0+h)-f(x_0)}{h}$. As h gets smaller, gradient of AB gets closer to gradient of curve at A.
Hence,
The gradient function or the derivative of a curve $y = f(x)$ is given by
$\boldsymbol{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$ $\lim_{h\to 0}$ means the limit as h tends to 0.
This rule is called **differentiating from the first principle**

Example 1: $f(x) = x^2$ a. Show that $f'(x) = \lim_{h\to 0}(2x + h)$ b. Hence deduce that $f'(x) = 2x$

a. Use the definition of derivative
$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

So $f(x) = x^2$ implies $f(x + h) = (x + h)^2$,
Substitute $f(x + h)$ and $f(x)$ into the above definition of derivative
$$f'(x) = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h}$$
Expand the bracket
$(x + h)^2 = (x + h)(x + h)$
$= x^2 + 2xh + h^2$
$$f'(x) = \lim_{h\to 0}\frac{x^2 + 2xh + h^2 - x^2}{h}$$
Factorise the numerator
↓
$$f'(x) = \lim_{h\to 0}\frac{2xh + h^2}{h} \implies f'(x) = \lim_{h\to 0}\frac{h(2x+h)}{h}$$
$$f'(x) = \lim_{h\to 0}(2x + h)$$

b.
$f'(x) = 2x$
From part a, we know that
$$f'(x) = \lim_{h\to 0}(2x + h)$$
Apply the limits
As h→0, $2x + h \to 2x$
h→0 means as h tends to 0 or h approaches to 0, $2x + h$ approaches $2x$
Hence,
$$f'(x) = \lim_{h\to 0}(2x + h) = 2x$$

Differentiating x^n

In this chapter you will only learn how to differentiate functions of the form x^n, where n is any number.
Use the following results to differentiate functions.

$y = f(x) = x^n$ then $\frac{dy}{dx} = f'(x) = nx^{n-1}$ where n is any real number

$y = f(x) = ax^n$ then $\frac{dy}{dx} = f'(x) = anx^{n-1}$ where n is any real number and a is a constant

Here is an example for you to understand how to apply the results
Example 2: Find the derivative, $f'(x)$ when $f(x)$ equals:
a. x^6 b. $10x^{-1}$

a. $f(x) = x^6$
You are supposed to find derivative of $f(x) = x^6$, use the result for x^n
So $f'(x) = 6x^{6-1} = 6x^5$

b. $f(x) = 10x^{-1}$
Use the result for ax^n
$f'(x) = 10 \times (-1) \times x^{-1-1} = -10\,x^{-2}$
$f'(x) = -10 \times \frac{1}{x^2} = -\frac{10}{x^2}$ As $x^{-2} = \frac{1}{x^2}$

Differentiating quadratics:
A quadratic function or a curve is given by $y = ax^2 + bx + c$, where a, b and c are constants.
The derivative of y is $\frac{dy}{dx} = 2ax + b$

Note: Differentiate each term one at a time
Derivative of only a constant term is always 0. So if $y = 2$ then $\frac{dy}{dx} = 0$

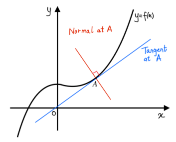
Example 3: Find the gradient of the curve with equation $y = 2x^2 - x - 1$ at the point (2,5)
As explained in the gradient of curves section, finding the gradient of a curve at a point is same as calculating derivative of the curve at that point.
So first, you will have to calculate the gradient of $y = 2x^2 - x - 1$

$\frac{dy}{dx} = 2 \times 2 \times x^{(2-1)} - 1 \times x^{(1-1)} - 0$ Since 1 is a constant, derivative of a constant will be 0
$\frac{dy}{dx} = 4x - x^0 = 4x - 1$ As $x^0 = 1$

Now to find the derivative of y at point(2,5), you need to substitute $x = 2$ in the derivative function
$$\frac{dy}{dx} = f'(2) = 4(2) - 1 = 8 - 1 = 7$$
Hence, the gradient of the curve with equation $y = 2x^2 - x - 1$ at the point (2,5) is 7.

Gradient, tangents and normal

In this section, you will learn to find equation of tangent and normal to a curve at a given point.
What is normal?
Normal to a curve at point A is a straight line passing through A and perpendicular to the tangent line at point A.
For a curve $y = f(x)$, the gradient of the tangent at point A with x coordinate a is $f'(a)$



The equation of tangent to the curve $y = f(x)$ at the point with coordinates $(a, f(a))$ is given by

$$y - f(a) = f'(a)(x - a)$$

You have come across in chapter 5 that the equation of straight line with gradient m that passes through the point (x_1, x_2) is given by
$$y - y_1 = m(x - x_2)$$

So, since the gradient of tangent at point A is $f'(a)$, the gradient of Normal at point A will be $-\frac{1}{f'(a)}$

The gradient of perpendicular lines are negative reciprocal of each other
i.e. $m_1 = -\frac{1}{m_2}$, where m_1 and m_2 are gradients

The equation of normal to the curve $y = f(x)$ at point $A \equiv (a, f(a))$ with gradient $-\frac{1}{f'(a)}$ is given by
$$y - f(a) = -\frac{1}{f'(a)}(x - a)$$
Here are some examples for you to understand how to find equation of tangent and normal.

Example 4: Find the equation of tangent and normal to the curve $y = x^2 - 7x + 10$ at the point (2,0)
First find the derivative of y, in order to find the gradient. Once you get the gradient function, substitute the x coordinate i.e. 2 into the function
So, $\frac{dy}{dx} = 2x - 7$
$\Rightarrow \frac{dy}{dx} = f'(2) = 2(2) - 7 = 4 - 7 = -3$
Substitute the gradient $\frac{dy}{dx} = -3$, $a = 2$ and $f(a) = 0$ into the equation of tangent
The equation of tangent is $y - f(a) = f'(a)(x - a)$
$\Rightarrow y - 0 = -3(x - 2) \Rightarrow y = -3x + 6$
Hence equation of tangent to the curve y at (2,0) is $y = -3x + 6$

Equation of normal:
Now as discussed earlier,
the gradient of normal is $-\frac{1}{f'(a)} = -\frac{1}{-3} = \frac{1}{3}$
So the equation of normal at point (2,0) will be,
$$y - f(a) = -\frac{1}{f'(a)}(x - a)$$

$\Rightarrow y - 0 = \frac{1}{3}(x - 2) \Rightarrow y = \frac{(x-2)}{3}$
Hence the equation of normal to the curve y at (2,0) is $y = \frac{(x-2)}{3}$

Increasing and decreasing function:
In this section you will be able to find out whether a function is increasing or decreasing.
The function $f(x)$ in the interval $[a, b]$, for all values of x where $a < b$ is

Increasing if	Decreasing if
$\boldsymbol{f'(x) \geq 0}$	$f'(x) \leq 0$

Second order derivative:
When you differentiate a function $y = f(x)$ once it called first order derivative i.e. $f'(x)$
And when you differentiate a function $f(x)$ twice, it called second order derivative and denoted by $f''(x)$ or $\frac{d^2y}{dx^2}$
Example 5: Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $y = 2x^2 + 7x - 3$

$\frac{dy}{dx} = (2 \times 2x^{2-1}) + 7 - 0 = 4x + 7$
To find $\frac{d^2y}{dx^2}$, find the derivative of $\frac{dy}{dx}$ i.e. $4x + 7$
So, $\frac{d^2y}{dx^2} = 4 \times x^0 + 0 = 4$ as $x^0 = 1$ and derivative of 7 is 0

Stationary points
A stationary point is any point on the curve where the gradient of the curve is 0. In this section you will learn to determine the nature of stationary points i.e. whether the stationary point is local maximum, a local minimum or a point of inflection (a point where the curve changes from being concave to convex or vice versa). You can refer to the table below,

Any point on the curve $y = f(x)$ where $f'(x) = 0$ is called a stationary point. For a small positive value h:

Type of stationary point	$f'(x-h)$	$f'(x)$	$f'(x+h)$
Local maximum	+ve (Positive)	0	-ve (Negative)
Local minimum	-ve (Negative)	0	+ve (Positive)
Point of Inflection	-ve (Negative)	0	-ve (Negative)
	+ve (Positive)	0	+ve (Positive)

You can also use second order derivative, $f''(x)$, to determine the nature of Stationary points

If a function $f(x)$ has a stationary point when x $= a$ ($i.e.\ f'(a) = 0$) then if:

$f''(a) > 0 \Rightarrow$ **Local minimum** $f''(a) < 0 \Rightarrow$ **Local maximum**	$f''(a) = 0 \longrightarrow$**Could be local minimum, local maximum or point of inflection**. You will have to use the above table to determine its nature.

Example 6: For $y = f(x) = x^3 - 3x^2 + 3x$, (1,1) is a stationary point.
Determine the nature of the stationary point.
You can use $f''(x)$ to find the nature of (1,1)
To find $f''(x)$, we need to find the derivative twice
Hence, $f'(x) = 3x^2 - 6x + 3$
$\Rightarrow f''(x) = 6x - 6$
Substituting $x = 1$
we get, $f''(1) = 6(1) - 6 = 0$

As $f''(0)$, you need to consider points on either side of $x = 1$,
i.e. you need to check sign or shape of gradient on either side of 1

x	0.9	1	1.1
$f'(x)$	0.03	0	0.03
Shape	/+ve	-----	/+ve

Since the gradient on both sides of (1,1) is positive, (1,1) is point of inflection

Example 7: a. Find the coordinates of the stationary point on $y = x^4 - 32x$
b. Determine the nature of stationary point using second order derivative
a. Find the derivate of y and equate to 0.
$\frac{dy}{dx} = 4x^3 - 32$
Let $\frac{dy}{dx} = 0$ and solve the equation to find the value of x
$4x^3 - 32 = 0 \implies 4x^3 = 32$
Hence, $x = 2$
Substituting the value of x in to the original equation we get y coordinate
So for $x = 2$, $y = 2^4 - 32 \times 2 = -48$
Hence $(2, -48)$ is a stationary point.

b. To find the nature of stationary point, find whether $f''(x) > 0$, $f''(x) < 0$ or $f''(x) = 0$
$\frac{d^2y}{dx^2} = 4 \times 3 \times x^{3-2} - 0 = 12x^2$
When $x = 2$, $\frac{d^2y}{dx^2} = 12(2)^2 = 48$
Since $f''(2) = 48 > 0$, point $(2, -48)$ is a local minimum

Modelling with differentiation:
In this section you will learn how to use derivatives to model real life situations involving rates of change.
$\frac{dy}{dx}$ represents the rate of change of y with respect to x. The term dy is small change in y and term dx is small change in x.
You know that speed is the change in distance over change in time. So if $s = f(t)$ is the function that represents distance of object from a fixed point at time t, then $\frac{ds}{dt} = f'(t)$ represents speed of the object at time t.

Example 8: Given that the volume, V cm³, of an expanding sphere is related to its radius, r cm, by the formula $V = \frac{4}{3}\pi r^3$, find the rate of change of volume with respect to radius at the instant when the radius is 5 cm.
As you know derivative represents rate of change, so in order to find rate of change of volume with respect to radius $r = 5$, you will have to find $\frac{dV}{dr}$
Hence, $\frac{dV}{dr} = \frac{4}{3} \times \pi \times 3 \times r^{3-1} = 4\pi r^2$ remember π is a constant, so it will be multiplied with 4
When $r = 5$, $\frac{dV}{dr} = 4\pi \times (5)^2 = 314$ remember π value is 3.14
So, the rate of change is 314 cm³ per cm. Interpret the answer with units $\left(\frac{dV}{dr} \to \frac{\text{cm}^3}{\text{cm}}\right)$